Chapter 5

Complex Numbers and Polar Coordinates

One of the goals of algebra is to find solutions to polynomial equations. You have probably done this many times in the past, solving equations like $x^2 - 1 = 0$ or $2x^2 + 1 = 3x$. In the process, you encountered the quadratic formula that allows us to find all solutions to quadratic equations. For example, the quadratic formula gives us the solutions $x = \frac{2 + \sqrt{-4}}{2}$ and $x = \frac{2 - \sqrt{-4}}{2}$ for the quadratic equation $x^2 - 2x + 2 = 0$. In this chapter we will make sense of solutions like these that involve negative numbers under square roots, and discover connections between algebra and trigonometry that will allow us to solve a larger collection of polynomial equations than we have been able to until now.

5.1 The Complex Number System

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is a complex number?
- What does it mean for two complex numbers to be equal?
- How do we add two complex numbers together?
- How do we multiply two complex numbers together?
- What is the conjugate of a complex number?
- What is the modulus of a complex number?
- How are the conjugate and modulus of a complex number related?
- How do we divide one complex number by another?

The quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ allows us to find solutions to the quadratic equation $ax^2 + bx + c = 0$. For example, the solutions to the equation $x^2 + x + 1 = 0$ are

$$x = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm \sqrt{-3}}{2}.$$

A problem arises immediately with this solution since there is no real number t with the property that $t^2 = -3$ or $t = \sqrt{-3}$. To make sense of solutions like this we introduce *complex numbers*. Although complex numbers arise naturally when solving quadratic equations, their introduction into mathematics came about from the problem of solving cubic equations.[1]

If we use the quadratic formula to solve an equation such as $x^2 + x + 1 = 0$,

[1] An interesting, and readable, telling of this history can be found in Chapter 6 of *Journey Through Genius* by William Dunham.

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we obtain the solutions $x = \dfrac{-1 + \sqrt{-3}}{2}$ and $x = \dfrac{-1 - \sqrt{-3}}{2}$. These numbers are complex numbers and we have a special form for writing these numbers. We write them in a way that that isolates the square root of -1. To illustrate, the number $\dfrac{-1 + \sqrt{-3}}{2}$ can be written as follows;

$$\begin{aligned}
\frac{-1 + \sqrt{-3}}{2} &= -\frac{1}{2} + \frac{\sqrt{-3}}{2} \\
&= -\frac{1}{2} + \frac{\sqrt{3}\sqrt{-1}}{2} \\
&= -\frac{1}{2} + \frac{\sqrt{3}}{2}\sqrt{-1}.
\end{aligned}$$

Since there is no real number t satisfying $t^2 = -1$, the number $\sqrt{-1}$ is not a real number. We call $\sqrt{-1}$ an *imaginary* number and give is a special label i. Thus, $i = \sqrt{-1}$ or $i^2 = -1$. With this in mind we can write

$$\frac{-1 + \sqrt{-3}}{2} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i,$$

and every complex number has this special form.

Definition. A **complex number** is an object of the form

$$a + bi,$$

where a and b are real numbers and $i^2 = -1$.

The form $a + bi$, where a and b are real numbers is called the **standard form** for a complex number. When we have a complex number of the form $z = a + bi$, the number a is called the **real part** of the complex number z and the number b is called the **imaginary part** of z. Since i is not a real number, two complex numbers $a + bi$ and $c + di$ are equal if and only if $a = c$ and $b = d$.

There is an arithmetic of complex numbers that is determined by an addition and multiplication of complex numbers. Adding and subtracting complex numbers is natural:

$$\begin{aligned}
(a + bi) + (c + di) &= (a + c) + (b + d)i \\
(a + bi) + (c + di) &= (a + c) + (b + d)i
\end{aligned}$$

That is, to add (or subtract) two complex numbers we add (subtract)their real parts and add (subract) their imaginary parts. Multiplication is also done in a natural way – to multiply two complex numbers, we simply expand the product as usual and exploit the fact that $i^2 = -1$. So the product of two complex number is

$$\begin{aligned}(a+bi)(c+di) &= ac + (ad)i + (bc)i + (bd)i^2 \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

It can be shown that the complex numbers satisfy many useful and familiar properties, which are similar to properties of the real numbers. If u, w, and z, are complex numbers, then

1. $w + z = z + w$
2. $u + (w + z) = (u + w) + z$
3. The complex number $0 = 0 + 0i$ is an additive identity, that is $z + 0 = z$.
4. If $z = a + bi$, then the additive inverse of z is $-z = (-a) + (-b)i$, That is, $z + (-z) = 0$.
5. $wz = zw$
6. $u(wz) = (uw)z$
7. $u(w + z) = uw + uz$
8. If $wz = 0$, then $w = 0$ or $z = 0$.

We will use these properties as needed. For example, to write the complex product $(1 + i)i$ in the form $a + bi$ with a and b real numbers, we distribute multiplication over addition and use the fact that $i^2 = -1$ to see that

$$(1+i)i = i + i^2 = i + (-1) = (-1) + i.$$

For another example, if $w = 2 + i$ and $z = 3 - 2i$, we can use these properties to write wz in the standard $a + bi$ form as follows:

$$\begin{aligned}wz &= (2+i)z \\ &= 2z + iz \\ &= 2(3-2i) + i(3-2i) \\ &= (6-4i) + (3i - 2i^2) \\ &= 6 - 4i + 3i - 2(-1) \\ &= 8 - i\end{aligned}$$

Progress Check 5.1 (Sums and Products of Complex Numbers)

1. Write each of the sums or products as a complex number in standard form.

   (a) $(2 + 3i) + (7 - 4i)$

   (b) $(4 - 2i)(3 + i)$

   (c) $(2 + i)i - (3 + 4i)$

2. Use the quadratic formula to write the two solutions to the quadratic equation $x^2 - x + 2 = 0$ as complex numbers of the form $r + si$ and $u + vi$ for some real numbers r, s, u, and v. (**Hint**: Remember: $i = \sqrt{-1}$. So we can rewrite something like $\sqrt{-4}$ as $\sqrt{-4} = \sqrt{4}\,\sqrt{-1} = 2i$.)

Division of Complex Numbers

We can add, subtract, and multiply complex numbers, so it is natural to ask if we can divide complex numbers. We illustrate with an example.

Example 5.2 (Dividing by a Complex Number)

Suppose we want to write the quotient $\dfrac{2+i}{3+i}$ as a complex number in the form $a + bi$. This problem is rationalizing a denominator since $i = \sqrt{-1}$. So in this case we need to "remove" the imaginary part from the denominator. Recall that the product of a complex number with its conjugate is a real number, so if we multiply the numerator and denominator of $\dfrac{2+i}{3+i}$ by the complex conjugate of the denominator, we can rewrite the denominator as a real number. The steps are as follows. Multiplying the numerator and denominator by the conjugate $3-i$ of $3+i$ gives us

$$
\begin{aligned}
\frac{2+i}{3+i} &= \left(\frac{2+i}{3+i}\right)\left(\frac{3-i}{3-i}\right) \\
&= \frac{(2+i)(3-i)}{(3+i)(3-i)} \\
&= \frac{(6-i^2) + (-2+3)i}{9-i^2} \\
&= \frac{7+i}{10}.
\end{aligned}
$$

Now we can write the final result in standard form as $\dfrac{7+i}{10} = \dfrac{7}{10} + \dfrac{1}{10}i$.

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Example 5.2 illustrates the general process for dividing one complex number by another. In general, we can write the quotient $\dfrac{a+bi}{c+di}$ in the form $r+si$ by multiplying numerator and denominator of our fraction by the conjugate $c-di$ of $c+di$ to see that

$$\begin{aligned}\frac{a+bi}{c+di} &= \left(\frac{a+bi}{c+di}\right)\left(\frac{c-di}{c-di}\right)\\ &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}\\ &= \frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i.\end{aligned}$$

Therefore, we have the formula for the quotient of two complex numbers.

The **quotient** $\dfrac{a+bi}{c+di}$ of the complex numbers $a+bi$ and $c+di$ is the complex number

$$\frac{a+bi}{c+di}=\frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i,$$

provided $c+di\neq 0$.

Progress Check 5.3 (Dividing Complex Numbers)
Let $z=3+4i$ and $w=5-i$.

1. Write $\dfrac{w}{z}=\dfrac{5-i}{3+4i}$ as a complex number in the form $r+si$ where r and s are some real numbers. Check the result by multiplying the quotient by $3+4i$. Is this product equal to $5-i$?
2. Find the solution to the equation $(3+4i)x=5-i$ as a complex number in the form $x=u+vi$ where u and v are some real numbers.

Geometric Representations of Complex Numbers

Each ordered pair (a,b) of real numbers determines:

- A point in the coordinate plane with coordinates (a,b).
- A complex number $a+bi$.
- A vector $a\mathbf{i}+b\mathbf{j}=\langle a,b\rangle$.

This means that we can geometrically represent the complex number $a + bi$ with a vector in standard position with terminal point (a, b). Therefore, we can draw pictures of complex numbers in the plane. When we do this, the horizontal axis is called **the real axis**, and the vertical axis is called **the imaginary axis**. In addition, the coordinate plane is then referred to as **the complex plane**. That is, if $z = a+bi$ we can think of z as a directed line segment from the origin to the point (a, b), where the terminal point of the segment is a units from the imaginary axis and b units from the real axis. For example, the complex numbers $3 + 4i$ and $-8 + 3i$ are shown in Figure 5.1.

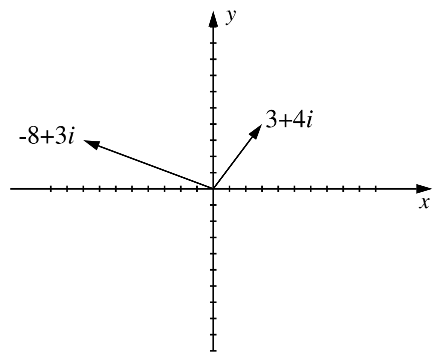

Figure 5.1: Two complex numbers.

In addition, the sum of two complex numbers can be represented geometrically using the vector forms of the complex numbers. Draw the parallelogram defined by $w = a+bi$ and $z = c+di$. The sum of w and z is the complex number represented by the vector from the origin to the vertex on the parallelogram opposite the origin as illustrated with the vectors $w = 3 + 4i$ and $z = -8 + 3i$ in Figure 5.2.

Progress Check 5.4 (Visualizing Complex Addition)
Let $w = 2 + 3i$ and $z = -1 + 5i$.

1. Write the complex sum $w + z$ in standard form.

2. Draw a picture to illustrate the sum using vectors to represent w and z.

We now extend our use of the representation of a complex number as a vector in standard position to include the notion of the length of a vector. Recall from Section 3.6 (page 234) that the length of a vector $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ is $|\mathbf{v}| = \sqrt{a^2 + b^2}$.

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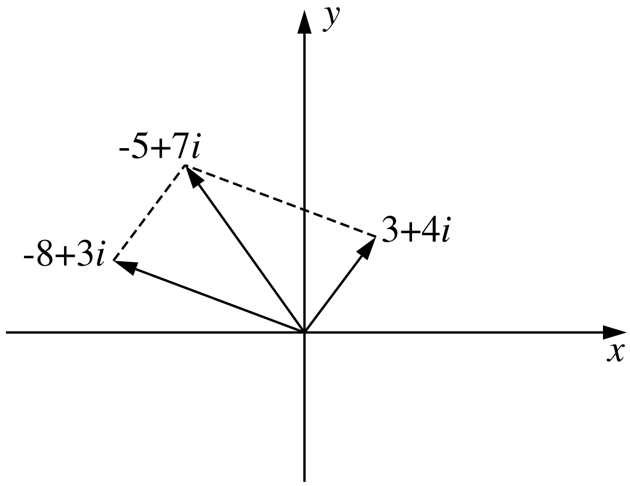

Figure 5.2: The Sum of Two Complex Numbers.

When we use this idea with complex numbers, we call it the *norm* or *modulus* of the complex number.

Definition. The **norm** (or **modulus**) of the complex number $z = a + bi$ is the distance from the origin to the point (a, b) and is denoted by $|z|$. We see that

$$|z| = |a + bi| = \sqrt{a^2 + b^2}.$$

There is another concept related to complex number that is based on the following bit of algebra.

$$\begin{aligned}(a + bi)(a - bi) &= a^2 - (bi)^2 \\ &= a^2 - b^2 i^2 \\ &= a^2 + b^2\end{aligned}$$

The complex number $a - bi$ is called the **complex conjugate** of $a + bi$. If we let $z = a + bi$, we denote the complex conjugate of z as $\overline{z}$. So

$$\overline{z} = \overline{a + bi} = a - bi.$$

We also notice that

$$z\overline{z} = (a + bi)(a - bi) = a^2 + b^2,$$

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and so the product of a complex number with its conjugate is a real number. In fact,

$$z\overline{z} = a^2 + b^2 = |z|^2, \text{ and so}$$
$$|z| = \sqrt{z\overline{z}}$$

Progress Check 5.5 (Operations on Complex Numbers)
Let $w = 2 + 3i$ and $z = -1 + 5i$.

1. Find $\overline{w}$ and $\overline{z}$.
2. Compute $|w|$ and $|z|$.
3. Compute $w\overline{w}$ and $z\overline{z}$.
4. What is $\overline{z}$ if z is a real number?

Summary of Section 5.1

In this section, we studied the following important concepts and ideas:

- A **complex number** is an object of the form $a + bi$, where a and b are real numbers and $i^2 = -1$. When we have a complex number of the form $z = a + bi$, the number a is called the **real part** of the complex number z and the number b is called the **imaginary part** of z.
- We can add, subtract, multiply, and divide complex numbers as follows:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi) + (c + di) = (a + c) + (b + d)i$$
$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$
$$\frac{a + bi}{c + di} = \frac{ac + bd}{c^2 + d^2} + \frac{bc - ad}{c^2 + d^2}i, \text{ provided } c + di \neq 0$$

- A complex number $a + bi$ can be represented geometrically with a vector in standard position with terminal point (a, b). When we do this, the horizontal axis is called **the real axis**, and the vertical axis is called **the imaginary axis**. In addition, the coordinate plane is then referred to as **the complex plane**. That is, if $z = a + bi$ we can think of z as a directed line segment from the origin to the point (a, b), where the terminal point of the segment is a units from the imaginary axis and b units from the real axis.

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- The **norm** (or **modulus**) of the complex number $z = a + bi$ is the distance from the origin to the point (a, b) and is denoted by $|z|$. We see that
$$|z| = |a + bi| = \sqrt{a^2 + b^2}.$$
- The complex number $a - bi$ is called the **complex conjugate** of $a + bi$. Note that
$$(a + bi)(a - bi) = a^2 + b^2 = |a + bi|^2.$$

Exercises for Section 5.1

⋆ **1.** Write each of the following as a complex number in standard form.

(a) $(4 + i) + (3 - 3i)$
(b) $5(2 - i) + i(3 - 2i)$
(c) $(4 + 2i)(5 - 3i)$
(d) $(2 + 3i)(1 + i) + (4 - 3i)$

2. Use the quadratic formula to write the two solutions of each of the following quadratic equations in standard form.

⋆ **(a)** $x^2 - 3x + 5 = 0$
(b) $2x^2 = x - 7$

3. For each of the following pairs of complex numbers w and z, determine the sum $w + z$ and illustrate the sum with a diagram.

⋆ **(a)** $w = 3 + 2i, z = 5 - 4i$.
⋆ **(b)** $w = 4i, z = -3 + 2i$.
(c) $w = 5, z = -7 + 2i$.
(d) $w = 6 - 3i, z = -1 + 7i$.

4. For each of the following complex numbers z, determine $\overline{z}$, $|z|$, and $z\overline{z}$.

⋆ **(a)** $z = 5 + 2i$
⋆ **(b)** $z = 3i$
(c) $z = 3 - 4i$
(d) $z = 7 + i$

5. Write each of the following quotients in standard form.

⋆ **(a)** $\dfrac{5 + i}{3 + 2i}$
⋆ **(b)** $\dfrac{3 + 3i}{i}$

(c) $\dfrac{i}{2-i}$ **(d)** $\dfrac{4+2i}{1-i}$

6. We know that $i^1 = i$ and $i^2 = -1$. We can then see that

$$i^3 = i^2 \cdot i = (-1)i = -i.$$

(**a**) Show that $i^4 = 1$.

(**b**) Now determine i^5, i^6, i^7, and i^8. **Note**: Each power of i will equal 1, -1, i, or $-i$.

(**c**) Notice that $13 = 4 \cdot 3 + 1$. We will use this to determine i^{13}.

$$i^{13} = i^{4 \cdot 3 + 1} = i^{4 \cdot 3} i^1 = \left(i^4\right)^3 \cdot i$$

So what is i^{13}?

(**d**) Using $39 = 4 \cdot 9 + 3$, determine i^{39}.

(**e**) Determine i^{54}.

7. (**a**) Write the complex number $i(2+2i)$ in standard form. Plot the complex numbers $2+2i$ and $i(2+2i)$ in the complex plane. What appears to be the angle between these two complex numbers?

(**b**) Repeat part (a) for the complex numbers $2-3i$ and $i(2-3i)$.

(**c**) Repeat part (a) for the complex numbers $3i$ and $i(3i)$.

(**d**) Describe what happens when the complex number $a+bi$ is multiplied by the complex number i.

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5.2 The Trigonometric Form of a Complex Number

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is the polar (trigonometric) form of a complex number?
- How do we multiply two complex numbers in polar form?
- How do we divide one complex number in polar form by a nonzero complex number in polar form?

Multiplication of complex numbers is more complicated than addition of complex numbers. To better understand the product of complex numbers, we first investigate the trigonometric (or polar) form of a complex number. This trigonometric form connects algebra to trigonometry and will be useful for quickly and easily finding powers and roots of complex numbers.

Beginning Activity

If $z = a + bi$ is a complex number, then we can plot z in the plane as shown in Figure 5.3. In this situation, we will let r be the magnitude of z (that is, the distance from z to the origin) and θ the angle z makes with the positive real axis as shown in Figure 5.3.

1. Use right triangle trigonometry to write a and b in terms of r and θ.
2. Explain why we can write z as

$$z = r(\cos(\theta) + i\,\sin(\theta)). \tag{1}$$

When we write z in the form given in Equation (1), we say that z is written in **trigonometric form** (or *polar form*).[2] The angle θ is called the **argument** of the

[2]The word *polar* here comes from the fact that this process can be viewed as occurring with polar coordinates.

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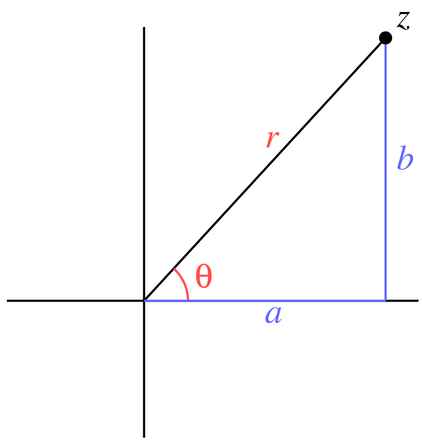

Figure 5.3: Trigonometric form of a complex number.

complex number z and the real number r is the **modulus** or **norm** of z. To find the polar representation of a complex number $z = a + bi$, we first notice that

$$\begin{aligned} r &= |z| = \sqrt{a^2 + b^2} \\ a &= r\cos(\theta) \\ b &= r\sin(\theta) \end{aligned}$$

To find θ, we have to consider cases.

- If $z = 0 = 0 + 0i$, then $r = 0$ and θ can have any real value.
- If $z \neq 0$ and $a \neq 0$, then $\tan(\theta) = \dfrac{b}{a}$.
- If $z \neq 0$ and $a = 0$ (so $b \neq 0$), then
  * $\theta = \dfrac{\pi}{2}$ if $b > 0$
  * $\theta = -\dfrac{\pi}{2}$ if $b < 0$.

Progress Check 5.6 (The Polar Form of a Complex Number)

1. Determine the polar form of the complex numbers $w = 4 + 4\sqrt{3}i$ and $z = 1 - i$.

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2. Determine real numbers a and b so that $a+bi = 3\left(\cos\left(\frac{\pi}{6}\right) + i\,\sin\left(\frac{\pi}{6}\right)\right)$.

There is an alternate representation that you will often see for the polar form of a complex number using a complex exponential. We won't go into the details, but only consider this as notation. When we write $e^{i\theta}$ (where i is the complex number with $i^2 = -1$) we mean

$$e^{i\theta} = \cos(\theta) + i\,\sin(\theta).$$

So the polar form $r(\cos(\theta) + i\,\sin(\theta))$ can also be written as $re^{i\theta}$:

$$re^{i\theta} = r(\cos(\theta) + i\,\sin(\theta)).$$

Products of Complex Numbers in Polar Form

There is an important product formula for complex numbers that the polar form provides. We illustrate with an example.

Example 5.7 (Products of Complex Numbers in Polar Form)

Let $w = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$ and $z = \sqrt{3} + i$. Using our definition of the product of complex numbers we see that

$$\begin{aligned} wz &= \left(\sqrt{3} + i\right)\left(-\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) \\ &= -\sqrt{3} + i. \end{aligned}$$

Now we write w and z in polar form. Note that $|w| = \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1$ and the argument of w satisfies $\tan(\theta) = -\sqrt{3}$. Since w is in the second quadrant, we see that $\theta = \frac{2\pi}{3}$, so the polar form of w is

$$w = \cos\left(\frac{2\pi}{3}\right) + i\,\sin\left(\frac{2\pi}{3}\right).$$

Also, $|z| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$ and the argument of z satisfies $\tan(\theta) = \frac{1}{\sqrt{3}}$. Since z is in the first quadrant, we know that $\theta = \frac{\pi}{6}$ and the polar form of z is

$$z = 2\left[\cos\left(\frac{\pi}{6}\right) + i\,\sin\left(\frac{\pi}{6}\right)\right].$$

We can also find the polar form of the complex product wz. Here we have $|wz| = 2$, and the argument of zw satisfies $\tan(\theta) = -\dfrac{1}{\sqrt{3}}$. Since wz is in quadrant II, we see that $\theta = \dfrac{5\pi}{6}$ and the polar form of wz is

$$wz = 2\left[\cos\left(\frac{5\pi}{6}\right) + i\,\sin\left(\frac{5\pi}{6}\right)\right].$$

When we compare the polar forms of w, z, and wz we might notice that $|wz| = |w|\,|z|$ and that the argument of zw is $\dfrac{2\pi}{3} + \dfrac{\pi}{6}$ or the sum of the arguments of w and z. This turns out to be true in general.

The result of Example 5.7 is no coincidence, as we will show. In general, we have the following important result about the product of two complex numbers.

Muliplication of Complex Numbers in Polar Form
Let $w = r(\cos(\alpha) + i\,\sin(\alpha))$ and $z = s(\cos(\beta) + i\,\sin(\beta))$ be complex numbers in polar form. Then the polar form of the complex product wz is given by

$$wz = rs\,(\cos(\alpha+\beta) + i\,\sin(\alpha+\beta))\,.$$

This states that to multiply two complex numbers in polar form, we multiply their norms and add their arguments.

To understand why this result it true in general, let $w = r(\cos(\alpha) + i\,\sin(\alpha))$ and $z = s(\cos(\beta) + i\,\sin(\beta))$ be complex numbers in polar form. We will use cosine and sine of sums of angles identities to find wz:

$$\begin{aligned} wz &= [r(\cos(\alpha) + i\,\sin(\alpha))][s(\cos(\beta) + i\,\sin(\beta))] \\ &= rs([\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)] + i[\cos(\alpha)\sin(\beta) + \cos(\beta)\sin(\alpha)] \end{aligned} \tag{1}$$

We now use the cosine and sum identities (see page 291) and see that

$$\begin{aligned} \cos(\alpha+\beta) &= \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta) \quad \text{and} \\ \sin(\alpha+\beta) &= \cos(\alpha)\sin(\beta) + \cos(\beta)\sin(\alpha) \end{aligned}$$

Using equation (1) and these identities, we see that

$$\begin{aligned} wz &= rs([\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)] + i[\cos(\alpha)\sin(\beta) + \cos(\beta)\sin(\alpha)] \\ &= rs(\cos(\alpha+\beta) + i\,\sin(\alpha+\beta)) \end{aligned}$$

as expected.

An illustration of this is given in Figure 5.4. The formula for multiplying complex numbers in polar form tells us that to multiply two complex numbers, we add their arguments and multiply their norms.

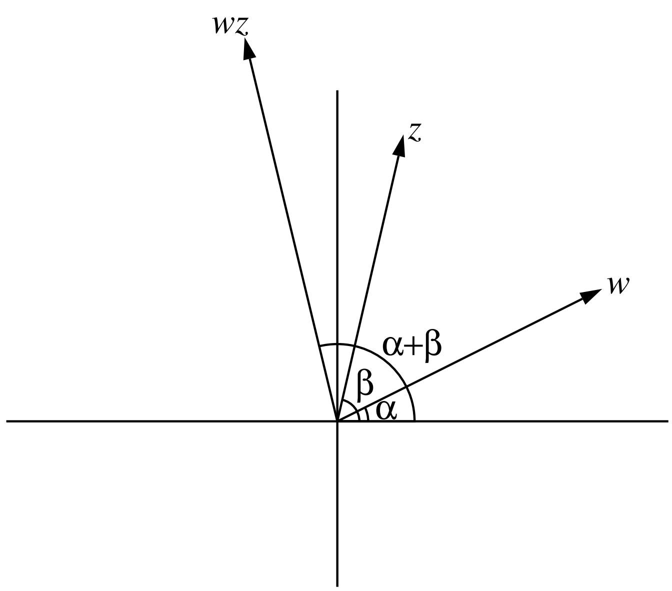

Figure 5.4: A Geometric Interpretation of Multiplication of Complex Numbers.

Progress Check 5.8 (Visualizing the Product of Complex Numbers)
Let $w = 3\left[\cos\left(\frac{5\pi}{3}\right) + i\,\sin\left(\frac{5\pi}{3}\right)\right]$ and $z = 2\left[\cos\left(-\frac{\pi}{4}\right) + i\,\sin\left(-\frac{\pi}{4}\right)\right]$.

1. What is $|wz|$?
2. What is the argument of wz?
3. In which quadrant is wz? Explain.
4. Determine the polar form of wz.
5. Draw a picture of w, z, and wz that illustrates the action of the complex product.

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Quotients of Complex Numbers in Polar Form

We have seen that we multiply complex numbers in polar form by multiplying their norms and adding their arguments. There is a similar method to divide one complex number in polar form by another complex number in polar form.

Division of Complex Numbers in Polar Form
Let $w = r(\cos(\alpha) + i\sin(\alpha))$ and $z = s(\cos(\beta) + i\sin(\beta))$ be complex numbers in polar form with $z \neq 0$. Then the polar form of the complex quotient $\dfrac{w}{z}$ is given by

$$\frac{w}{z} = \frac{r}{s}\left(\cos(\alpha - \beta) + i\sin(\alpha - \beta)\right).$$

So to divide complex numbers in polar form, we divide the norm of the complex number in the numerator by the norm of the complex number in the denominator and subtract the argument of the complex number in the denominator from the argument of the complex number in the numerator.

The proof of this is similar to the proof for multiplying complex numbers and is included as a supplement to this section.

Progress Check 5.9 (Visualizing the Quotient of Two Complex Numbers)
Let $w = 3\left[\cos\left(\dfrac{5\pi}{3}\right) + i\sin\left(\dfrac{5\pi}{3}\right)\right]$ and $z = 2\left[\cos\left(-\dfrac{\pi}{4}\right) + i\sin\left(-\dfrac{\pi}{4}\right)\right]$.

1. What is $\left|\dfrac{w}{z}\right|$?

2. What is the argument of $\left|\dfrac{w}{z}\right|$?

3. In which quadrant is $\left|\dfrac{w}{z}\right|$? Explain.

4. Determine the polar form of $\left|\dfrac{w}{z}\right|$.

5. Draw a picture of w, z, and $\left|\dfrac{w}{z}\right|$ that illustrates the action of the complex product.

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Proof of the Rule for Dividing Complex Numbers in Polar Form

Let $w = r(\cos(\alpha) + i\sin(\alpha))$ and $z = s(\cos(\beta) + i\sin(\beta))$ be complex numbers in polar form with $z \neq 0$. So

$$\frac{w}{z} = \frac{r(\cos(\alpha) + i\sin(\alpha))}{s(\cos(\beta) + i\sin(\beta))} = \frac{r}{s}\left[\frac{\cos(\alpha) + i\sin(\alpha)}{\cos(\beta) + i\sin(\beta)}\right].$$

We will work with the fraction $\dfrac{\cos(\alpha) + i\sin(\alpha)}{\cos(\beta) + i\sin(\beta)}$ and follow the usual practice of multiplying the numerator and denominator by $\cos(\beta) - i\sin(\beta)$. So

$$\begin{aligned}
\frac{w}{z} &= \frac{r}{s}\left[\frac{\cos(\alpha) + i\sin(\alpha)}{\cos(\beta) + i\sin(\beta)}\right] \\
&= \frac{r}{s}\left[\frac{\cos(\alpha) + i\sin(\alpha)}{\cos(\beta) + i\sin(\beta)} \cdot \frac{\cos(\beta) - i\sin(\beta)}{\cos(\beta) - i\sin(\beta)}\right] \\
&= \frac{r}{s}\left[\frac{(\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)) + i(\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta))}{\cos^2(\beta) + \sin^2(\beta)}\right]
\end{aligned}$$

We now use the following identities with the last equation:

- $\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta) = \cos(\alpha - \beta)$.
- $\sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta) = \sin(\alpha - \beta)$.
- $\cos^2(\beta) + \sin^2(\beta) = 1$.

Using these identities with the last equation for $\dfrac{w}{z}$, we see that

$$\frac{w}{z} = \frac{r}{s}\left[\frac{\cos(\alpha - \beta) + i\sin(\alpha - \beta)}{1}\right] = \frac{r}{s}[\cos(\alpha - \beta) + i\sin(\alpha - \beta)].$$

Summary of Section 5.2

In this section, we studied the following important concepts and ideas:

- If $z = a + bi$ is a complex number, then we can plot z in the plane. If r is the magnitude of z (that is, the distance from z to the origin) and θ the angle z makes with the positive real axis, then the **trigonomtric form** (or **polar form**) of z is $z = r(\cos(\theta) + i\sin(theta))$, where

  $$r = \sqrt{a^2 = b^2}, \cos(\theta) = \frac{a}{r}, \text{ and } \sin(\theta) = \frac{b}{r}.$$

  The angle θ is called the **argument** of the complex number z and the real number r is the **modulus** or **norm** of z.

- If $w = r(\cos(\alpha) + i\sin(\alpha))$ and $z = s(\cos(\beta) + i\sin(\beta))$ are complex numbers in polar form, then the polar form of the complex product wz is given by
$$wz = rs\,(\cos(\alpha + \beta) + i\sin(\alpha + \beta)),$$
and f $z \neq 0$, the polar form of the complex quotient $\dfrac{w}{z}$ is
$$\frac{w}{z} = \frac{r}{s}\,(\cos(\alpha - \beta) + i\sin(\alpha - \beta)),$$
This states that to multiply two complex numbers in polar form, we multiply their norms and add their arguments, and to divide two complex numbers, we divide their norms and subtract their arguments.

Exercises for Section 5.2

1. Determine the polar (trigonometric) form of each of the following complex numbers.

* **(a)** $3 + 3i$
(b) $3 - 3i$
(c) $-3 + 3i$
(d) $5i$
* **(e)** $4\sqrt{3} + 4i$
(f) $-4\sqrt{3} - 4i$

2. In each of the following a complex number z is given. In each case, determine real numbers a and b so that $z = a + bi$. If it is not possible to determine exact values for a and b, determine the values of a and b correct to four decimal places.

* **(a)** $z = 5\left(\cos\left(\dfrac{\pi}{2}\right) + i\sin\left(\dfrac{\pi}{2}\right)\right)$

* **(b)** $z = 2.5\left(\cos\left(\dfrac{\pi}{4}\right) + i\sin\left(\dfrac{\pi}{4}\right)\right)$

(c) $z = 2.5\left(\cos\left(\dfrac{3\pi}{4}\right) + i\sin\left(\dfrac{3\pi}{4}\right)\right)$

(d) $z = 3\left(\cos\left(\dfrac{7\pi}{6}\right) + i\sin\left(\dfrac{7\pi}{6}\right)\right)$

(e) $z = 8\left(\cos\left(\dfrac{7\pi}{10}\right) + i\sin\left(\dfrac{7\pi}{10}\right)\right)$

3. For each of the following, write the product wz in polar (trigonometric form). When it is possible, write the product in form $a + bi$, where a and b are real numbers and do not involve a trigonometric function.

⋆ **(a)** $w = 5\left(\cos\left(\dfrac{\pi}{12}\right) + i\,\sin\left(\dfrac{\pi}{12}\right)\right),\ z = 2\left(\cos\left(\dfrac{5\pi}{12}\right) + i\,\sin\left(\dfrac{5\pi}{12}\right)\right)$

⋆ **(b)** $w = 2.3\left(\cos\left(\dfrac{\pi}{3}\right) + i\,\sin\left(\dfrac{\pi}{3}\right)\right),\ z = 3\left(\cos\left(\dfrac{5\pi}{4}\right) + i\,\sin\left(\dfrac{5\pi}{4}\right)\right)$

(c) $w = 2\left(\cos\left(\dfrac{7\pi}{10}\right) + i\,\sin\left(\dfrac{7\pi}{10}\right)\right),\ z = 2\left(\cos\left(\dfrac{2\pi}{5}\right) + i\,\sin\left(\dfrac{2\pi}{5}\right)\right)$

(d) $w = (\cos(24^\circ) + i\,\sin(24^\circ)),\ z = 2(\cos(33^\circ) + i\,\sin(33^\circ))$

(e) $w = 2(\cos(72^\circ) + i\,\sin(72^\circ)),\ z = 2(\cos(78^\circ) + i\,\sin(78^\circ))$

⋆ **4.** For the complex numbers in Exercise (3), write the quotient $\dfrac{w}{z}$ in polar (trigonometric form). When it is possible, write the quotient in form $a + bi$, where a and b are real numbers and do not involve a trigonometric function.

5. (**a**) Write the complex number i in polar form.

(**b**) Let $z = r(\cos(\theta) + i\,\sin(\theta))$. Determine the product $i \cdot z$ in polar form. Use this to explain why the complex number $i \cdot z$ and z will be perpendicualr when both are plotted in the complex plane.

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5.3 DeMoivre's Theorem and Powers of Complex Numbers

Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- What is de Moivre's Theorem and why is it useful?
- If n is a positive integer, what is an nth root of a complex number? How many nth roots does a complex number have? How do we find all of the nth roots of a complex number?

The trigonometric form of a complex number provides a relatively quick and easy way to compute products of complex numbers. As a consequence, we will be able to quickly calculate powers of complex numbers, and even roots of complex numbers.

Beginning Activity

Let $z = r(\cos(\theta) + i\,\sin(\theta))$. Use the trigonometric form of z to show that

$$z^2 = r^2\left(\cos(2\theta) + i\,\sin(2\theta)\right). \tag{1}$$

De Moivre's Theorem

The result of Equation (1) is not restricted to only squares of a complex number. If $z = r(\cos(\theta) + i\,\sin(\theta))$, then it is also true that

$$\begin{aligned} z^3 &= zz^2 \\ &= (r)(r^2)\left(\cos(\theta + 2\theta) + i\,\sin(\theta + 2\theta)\right) \\ &= r^3\left(\cos(3\theta) + i\,\sin(3\theta)\right). \end{aligned}$$

We can continue this pattern to see that

$$\begin{aligned} z^4 &= zz^3 \\ &= (r)(r^3)\left(\cos(\theta + 3\theta) + i\,\sin(\theta + 3\theta)\right) \\ &= r^4\left(\cos(4\theta) + i\,\sin(4\theta)\right). \end{aligned}$$

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The equations for z^2, z^3, and z^4 establish a pattern that is true in general. The result is called de Moivre's Theorem.

DeMoivre's Theorem
Let $z = r(\cos(\theta) + i\,\sin(\theta))$ be a complex number and n any integer. Then

$$z^n = r^n(\cos(n\theta) + i\,\sin(n\theta)).$$

It turns out that DeMoivre's Theorem also works for negative integer powers as well.

Progress Check 5.10 (DeMoivre's Theorem)
Write the complex number $1 - i$ in polar form. Then use DeMoivre's Theorem to write $(1-i)^{10}$ in the complex form $a + bi$, where a and b are real numbers and do not involve the use of a trigonometric function.

Roots of Complex Numbers

DeMoivre's Theorem is very useful in calculating powers of complex numbers, even fractional powers. We illustrate with an example.

Example 5.11 (Roots of Complex Numbers)
We will find all of the solutions to the equation $x^3 - 1 = 0$. These solutions are also called the *roots* of the polynomial $x^3 - 1$. To solve the equation $x^3 - 1 = 0$, we add 1 to both sides to rewrite the equation in the form $x^3 = 1$. Recall that to solve a polynomial equation like $x^3 = 1$ means to find all of the numbers (real or complex) that satisfy the equation. We can take the real cube root of both sides of this equation to obtain the solution $x_0 = 1$, but every cubic polynomial should have three solutions. How can we find the other two? If we draw the graph of $y = x^3 - 1$ we see that the graph intersects the x-axis at only one point, so there is only one real solution to $x^3 = 1$. That means the other two solutions must be complex and we can use DeMoivre's Theorem to find them. To do this, suppose

$$z = r\,[\cos(\theta) + i\,\sin(\theta)]$$

is a solution to $x^3 = 1$. Then

$$1 = z^3 = r^3(\cos(3\theta) + i\,\sin(3\theta)).$$

This implies that $r = 1$ (or $r = -1$, but we can incorporate the latter case into our choice of angle). We then reduce the equation $x^3 = 1$ to the equation

$$1 = \cos(3\theta) + i\,\sin(3\theta). \tag{2}$$

Equation (2) has solutions when $\cos(3\theta) = 1$ and $\sin(3\theta) = 0$. This will occur when $3\theta = 2\pi k$, or $\theta = \dfrac{2\pi k}{3}$, where k is any integer. The distinct integer multiples of $\dfrac{2\pi k}{3}$ on the unit circle occur when $k = 0$ and $\theta = 0$, $k = 1$ and $\theta = \dfrac{2\pi}{3}$, and $k = 2$ with $\theta = \dfrac{4\pi}{3}$. In other words, the solutions to $x^3 = 1$ should be

$$\begin{aligned}
x_0 &= \cos(0) + i\,\sin(0) = 1\\
x_1 &= \cos\left(\frac{2\pi}{3}\right) + i\,\sin\left(\frac{2\pi}{3}\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i\\
x_2 &= \cos\left(\frac{4\pi}{3}\right) + i\,\sin\left(\frac{4\pi}{3}\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i.
\end{aligned}$$

We already know that $x_0^3 = 1^3 = 1$, so x_0 actually is a solution to $x^3 = 1$. To check that x_1 and x_2 are also solutions to $x^3 = 1$, we apply DeMoivre's Theorem:

$$\begin{aligned}
x_1^3 &= \left[\cos\left(\frac{2\pi}{3}\right) + i\,\sin\left(\frac{2\pi}{3}\right)\right]^3\\
&= \cos\left(3\left(\frac{2\pi}{3}\right)\right) + i\,\sin\left(3\left(\frac{2\pi}{3}\right)\right)\\
&= \cos(2\pi) + i\,\sin(2\pi)\\
&= 1,
\end{aligned}$$

and

$$\begin{aligned}
x_2^3 &= \left[\cos\left(\frac{4\pi}{3}\right) + i\,\sin\left(\frac{4\pi}{3}\right)\right]^3\\
&= \cos\left(3\left(\frac{4\pi}{3}\right)\right) + i\,\sin\left(3\left(\frac{4\pi}{3}\right)\right)\\
&= \cos(4\pi) + i\,\sin(4\pi)\\
&= 1.
\end{aligned}$$

Thus, $x_1^3 = 1$ and $x_2^3 = 1$ and we have found three solutions to the equation $x^3 = 1$. Since a cubic can have only three solutions, we have found them all.

The general process of solving an equation of the form $x^n = a + bi$, where n is a positive integer and $a + bi$ is a complex number works the same way. Write $a + bi$ in trigonometric form

$$a + bi = r\,[\cos(\theta) + i\,\sin(\theta)]\,,$$

and suppose that $z = s\,[\cos(\alpha) + i\,\sin(\alpha)]$ is a solution to $x^n = a + bi$. Then

$$\begin{aligned} a + bi &= z^n \\ r\,[\cos(\theta) + i\,\sin(\theta)] &= (s\,[\cos(\alpha) + i\,\sin(\alpha)])^n \\ r\,[\cos(\theta) + i\,\sin(\theta)] &= s^n\,[\cos(n\alpha) + i\,\sin(n\alpha)] \end{aligned}$$

Using the last equation, we see that

$$s^n = r \quad \text{and} \quad \cos(\theta) + i\,\sin(\theta) = \cos(n\alpha) + i\,\sin(n\alpha).$$

Therefore,

$$s^n = r \quad \text{and} \quad n\alpha = \theta + 2\pi k$$

where k is any integer. This give us

$$s = \sqrt[n]{r} \quad \text{and} \quad \alpha = \frac{\theta + 2\pi k}{n}.$$

We will get n different solutions for $k = 0, 1, 2, \ldots, n - 1$, and these will be all of the solutions. These solutions are called the nth roots of the complex number $a + bi$. We summarize the results.

Roots of Complex Numbers

Let n be a positive integer. The nth roots of the complex number $r\,[\cos(\theta) + i\,\sin(\theta)]$ are given by

$$\sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\,\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, (n - 1)$.

If we want to represent the nth roots of $r\,[\cos(\theta) + i\,\sin(\theta)]$ using degrees instead of radians, the roots will have the form

$$\sqrt[n]{r}\left[\cos\left(\frac{\theta + 360^\circ k}{n}\right) + i\,\sin\left(\frac{\theta + 360^\circ k}{n}\right)\right]$$

for $k = 0, 1, 2, \ldots, (n - 1)$.

Example 5.12 (Square Roots of 1)

As another example, we find the complex square roots of 1. In other words, we find the solutions to the equation $z^2 = 1$. Of course, we already know that the square roots of 1 are 1 and -1, but it will be instructive to utilize our general result and see that it gives the same result. Note that the trigonometric form of 1 is

$$1 = \cos(0) + i\,\sin(0),$$

so the two square roots of 1 are

$$\sqrt{1}\left[\cos\left(\frac{0+2\pi(0)}{2}\right)+i\,\sin\left(\frac{0+2\pi(0)}{2}\right)\right]=\cos(0)+i\,\sin(0)=1$$

and

$$\sqrt{1}\left[\cos\left(\frac{0+2\pi(1)}{2}\right)+i\,\sin\left(\frac{0+2\pi(1)}{2}\right)\right]=\cos(pi)+i\,\sin(\pi)=-1$$

as expected.

Progress Check 5.13 (Roots of Unity)

1. Find all solutions to $x^4 = 1$. (The solutions to $x^n = 1$ are called the nth roots of unity, with unity being the number 1.)
2. Find all sixth roots of unity.

Now let's apply our result to find roots of complex numbers other than 1.

Example 5.14 (Roots of Other Complex Numbers)
We will find the solutions to the equation

$$x^4 = -8 + 8\sqrt{3}i.$$

Note that we can write the right hand side of this equation in trigonometric form as

$$-8+8\sqrt{3}i = 16\left(\cos\left(\frac{2\pi}{3}\right)+i\,\sin\left(\frac{2\pi}{3}\right)\right).$$

The fourth roots of $-8+8\sqrt{3}i$ are then

$$\begin{aligned}x_0 &= \sqrt[4]{16}\left[\cos\left(\frac{\frac{2\pi}{3}+2\pi(0)}{4}\right)+i\,\sin\left(\frac{\frac{2\pi}{3}+2\pi(0)}{4}\right)\right]\\ &= 2\left[\cos\left(\frac{\pi}{6}\right)+i\,\sin\left(\frac{\pi}{6}\right)\right]\\ &= 2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)\\ &= \sqrt{3}+i,\end{aligned}$$

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$$
\begin{aligned}
x_1 &= \sqrt[4]{16}\left[\cos\left(\frac{\frac{2\pi}{3}+2\pi(1)}{4}\right)+i\sin\left(\frac{\frac{2\pi}{3}+2\pi(1)}{4}\right)\right] \\
&= 2\left[\cos\left(\frac{2\pi}{3}\right)+i\sin\left(\frac{2\pi}{3}\right)\right] \\
&= 2\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) \\
&= -1+\sqrt{3}i,
\end{aligned}
$$

$$
\begin{aligned}
x_2 &= \sqrt[4]{16}\left[\cos\left(\frac{\frac{2\pi}{3}+2\pi(2)}{4}\right)+i\sin\left(\frac{\frac{2\pi}{3}+2\pi(2)}{4}\right)\right] \\
&= 2\left[\cos\left(\frac{7\pi}{6}\right)+i\sin\left(\frac{7\pi}{6}\right)\right] \\
&= 2\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right) \\
&= -\sqrt{3}-i,
\end{aligned}
$$

and

$$
\begin{aligned}
x_3 &= \sqrt[4]{16}\left[\cos\left(\frac{\frac{2\pi}{3}+2\pi(3)}{4}\right)+i\sin\left(\frac{\frac{2\pi}{3}+2\pi(3)}{4}\right)\right] \\
&= 2\left[\cos\left(\frac{5\pi}{3}\right)+i\sin\left(\frac{5\pi}{3}\right)\right] \\
&= 2\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right) \\
&= 1-\sqrt{3}i.
\end{aligned}
$$

Progress Check 5.15 (Fourth Roots of −256)
Find all fourth roots of −256, that is find all solutions of the equation $x^4 = -256$.

Summary of Section 5.3

In this section, we studied the following important concepts and ideas:

- **DeMoivre's Theorem**. Let $z = r(\cos(\theta) + i\,\sin(\theta))$ be a complex number and n any integer. Then
$$z^n = r^n(\cos(n\theta) + i\,\sin(n\theta)).$$
- **Roots of Complex Numbers**. Let n be a positive integer. The nth roots of the complex number $r\,[\cos(\theta) + i\,\sin(\theta)]$ are given by
$$\sqrt[n]{r}\left[\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\,\sin\left(\frac{\theta + 2\pi k}{n}\right)\right]$$
for $k = 0, 1, 2, \ldots, (n-1)$.

Exercises for Section 5.3

1. Use DeMoivre's Theorem to determine each of the following powers of a complex number. Write the answer in the form $a + bi$, where a and b are real numbers and do not involve the use of a trigonometric function.

$\star$ **(a)** $(2 + 2i)^6$

$\star$ **(b)** $\left(\sqrt{3} + i\right)^8$

(c) $\left(\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i\right)^3$

(d) $2\left(\cos\left(\dfrac{\pi}{15}\right) + i\,\sin\left(\dfrac{\pi}{15}\right)\right)^{10}$

(e) $\left(1 + i\sqrt{3}\right)^{-4}$

(f) $(-3 + 3i)^{-3}$

2. In each of the following, determine the indicated roots of the given complex number. When it is possible, write the roots in the form $a + bi$, where a and b are real numbers and do not involve the use of a trigonometric function. Otherwise, leave the roots in polar form.

$\star$ **(a)** The two square roots of $16i$.

(b) The two square roots of $2 + 2i\sqrt{3}$.

$\star$ **(c)** The three cube roots of $5\left(\cos\left(\dfrac{3\pi}{4}\right) + i\,\sin\left(\dfrac{3\pi}{4}\right)\right)$.

(d) The five fifth roots of unity.

(e) The four fourth roots of $\left(\dfrac{1}{2} - \dfrac{\sqrt{3}}{2}i\right)$.

(f) The three cube roots of $1 + \sqrt{3}i$.

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5.4 The Polar Coordinate System

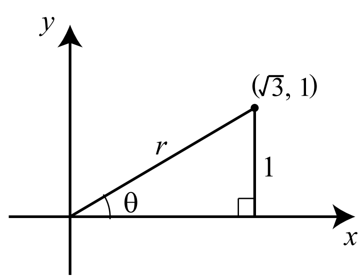
Focus Questions

The following questions are meant to guide our study of the material in this section. After studying this section, we should understand the concepts motivated by these questions and be able to write precise, coherent answers to these questions.

- How are the polar coordinates of a point in the plane determined?
- How do we convert from polar coordinates to rectangular coordinates?
- How do we convert from rectangular to polar coordinates?
- How doe we correctly graph polar equations both by hand and with a calculator?

Beginning Activity

In the diagram to the right, the point with coordinates $(\sqrt{3}, 1)$ has been plotted. Determine the value of r and the angle θ in radians and degrees.



Introduction

In our study of trigonometry so far, whenever we graphed an equation or located a point in the plane, we have used rectangular (or Cartesian [3]) coordinates. The use of this type of coordinate system revolutionized mathematics since it provided the first systematic link between geometry and algebra. Even though the rectangular coordinate system is very important, there are other methods of locating points in the plane. We will study one such system in this section.

Rectangular coordinates use two numbers (in the form of an ordered pair) to determine the loction of a point in the plane. These numbers give the position of a

[3]Named after the 17^{th} century mathematician, René Descartes)

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point relative to a pair of perpendicular axes. In the beginning activity, to reach the point that corresponds to the ordered pair $\left(\sqrt{3}, 1\right)$, we start at the origin and travel $\sqrt{3}$ units to the right and then travel 1 unit up. The idea of the polar coordinate system is to give a distance to travel and an angle in which direction to travel. We reach the same point as the one given by the rectangular coordinates $\left(\sqrt{3}, 1\right)$ by saying we will travel 2 units at an angle of $30°$ from the x-axis. These values correspond to the values of r and θ in the diagram for the beginning activity. Using the Pythagorean Theorem, we can obtain $r = 2$ and using the fact that $\sin(\theta) = \dfrac{1}{2}$, we see that $\theta = \dfrac{\pi}{6}$ radians or $30°$.

The Polar Coordinate System

For the rectangular coordinate system, we use two numbers, in the form of an ordered pair, to locate a point in the plane. We do the same thing for polar coordinates, but now the first number represents a distance from a point and the second number represents an angle. In the **polar coordinate system**, we start with a point O, called the **pole** and from this point, we draw a horizontal ray (directed half-line) called the **polar axis**. We can then assign polar coordinates (r, θ) to a point P in the plane as follows (see Figure 5.5):

- The number r, called the **radial distance**, is the directed distance from the pole to the point P.
- The number θ, called the **polar angle**, is the measure of the angle from the polar axis to the line segment OP. (Either radians or degrees can be used for the measure of the angle.)

Conventions for Polar Coordinates

- The polar angle θ is considered positive if measured in a counterclockwise direction from the polar axis.
- The polar angle θ is considered negative if measured in a clockwise direction from the polar axis.
- If the radial distance r is positive, then the point P is r units from O along the terminal side of θ.

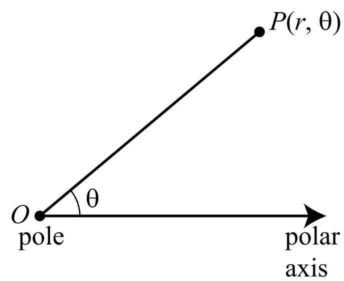

Figure 5.5: Polar Coordinates

- If the radial distance r is negative, then the point P is $|r|$ units from O along the ray in the opposite direction as the terminal side of θ.
- If the radial distance r is zero, then the point P is the point O.

To illustrate some of these conventions, consider the point $P\left(3, \dfrac{4\pi}{3}\right)$ shown on the left in Figure 5.6. (Notice that the circle of radius 3 with center at the pole has been drawn.)

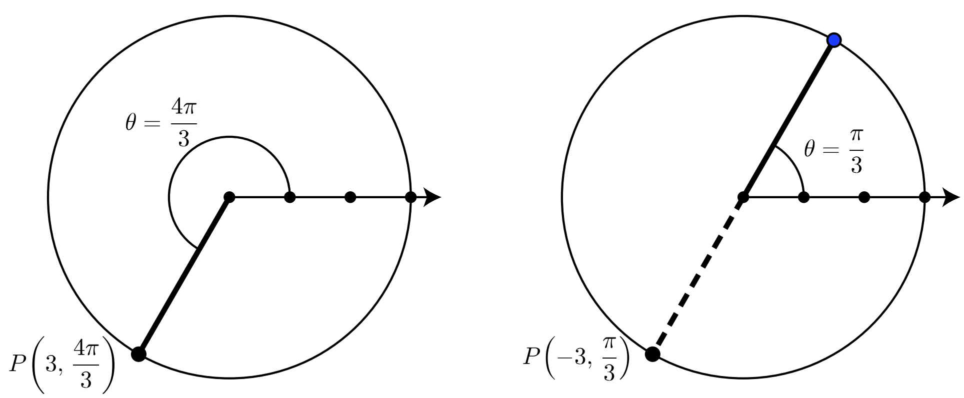

Figure 5.6: A Point with Two Different Sets of Polar Coordinates

The diagram on the right in Figure 5.6 illustrates that this point P also has polar coordinates $P\left(-3, \dfrac{\pi}{3}\right)$. This is because when we use the polar angle $\theta = \dfrac{\pi}{3}$ and the radial distance $r = -3$, then the point P is 3 units from the pole along the ray in the opposite direction as the terminal side of θ.

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Progress Check 5.16 (Plotting Points in Polar Coordinates)
Since a point with polar coordinates (r, θ) must lie on a circle of radius r with center at the pole, it is reasonable to locate points on a grid of concentric circles and rays whose initial point is at the pole as shown in Figure 5.7. On this polar graph paper, each angle increment is $\dfrac{\pi}{12}$ radians. For example, the point $\left(4, \dfrac{\pi}{6}\right)$ is plotted in Figure 5.7.

Plot the following points with the specified polar coordinates.

$\left(1, \dfrac{\pi}{4}\right)$	$\left(5, \dfrac{\pi}{4}\right)$	$\left(2, \dfrac{\pi}{3}\right)$
$\left(3, \dfrac{5\pi}{4}\right)$	$\left(4, -\dfrac{\pi}{4}\right)$	$\left(4, \dfrac{7\pi}{4}\right)$
$\left(6, \dfrac{5\pi}{6}\right)$	$\left(5, \dfrac{9\pi}{4}\right)$	$\left(-5, \dfrac{5\pi}{4}\right)$

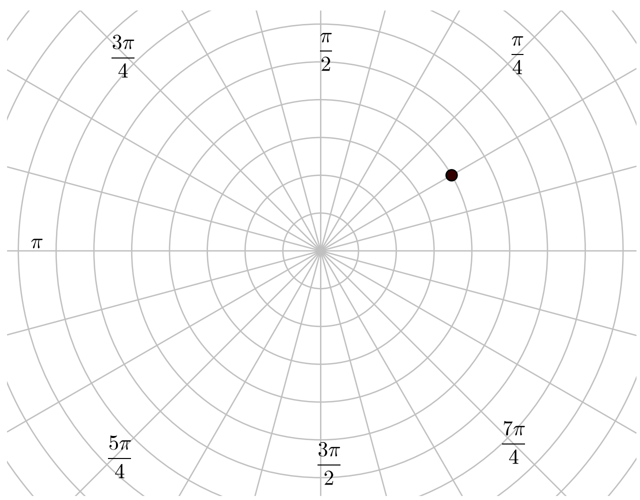

Figure 5.7: Polar Graph Paper

In Progress Check 5.16, we noticed that the polar coordinates $\left(5, \dfrac{\pi}{4}\right)$, $\left(5, \dfrac{9\pi}{4}\right)$, and $\left(-5, \dfrac{5\pi}{4}\right)$ all determined the same point in the plane. This illustrates a major difference between rectangular coordinates and polar coordinates. Whereas each point has a unique representation in rectangular coordinates, a given point can have

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many different representations in polar coordinates. This is primarily due to the fact that the polar coordinate system uses concentric cirlces for its grid, and we can start at a point on a circle and travel around the circle and end at the point from which we started. Since one wrap around a circle corresponds to an angle of 2π radians or $360°$, we have the following:

Polar Coordinates of a Point
A point P, other than the pole, determined by the polar coordinates (r, θ) is also determined by the following polar coordinates:

$$\begin{array}{lcc} \text{In radians :} & (r, \theta + k(2\pi)) & (-r, +\theta + (2k+1)\pi) \\ \text{In degrees :} & \left(r, \theta + k\left(360°\right)\right) & \left(-r, \theta + (2k+1)180°\right) \end{array}$$

where k can be any integer.
If the point P is the pole, the its polar coordinates are $(0, \theta)$ for any polar angle θ.

Progress Check 5.17 (Different Polar Coordinates for a Point)
Find four different representations in polar coordinates for the point with polar coordinates $(3, 110°)$. Use a positive value for the radial distance r for two of the representations and a negative value for the radial distance r for the other two representations.

Conversions Between Polar and Rectangular Coordinates

We now have two ways to locate points in the plane. One is the usual rectangular (Cartesian) coordinate system and the other is the polar coordinate system. The rectangular coordinate system uses two distances to locate a point, whereas the polar coordinate system uses a distance and an angle to locate a point. Although these two systems can be studied indpendently of each other, we can set them up so that there is a relationship between the two types of coordinates. We do this as follows:

- We place the pole of the polar coordinate system at the origin of the rectangular coordinate system.
- We have the polar axis of the polar coordinate system coincide with the positive x-axis of the rectangular coordinate system as shown in Figure 5.8

Using right triangle trigonometry and the Pythagorean Theorem, we obtain the following relationships between the rectangular coordinates (x, y) and the polar

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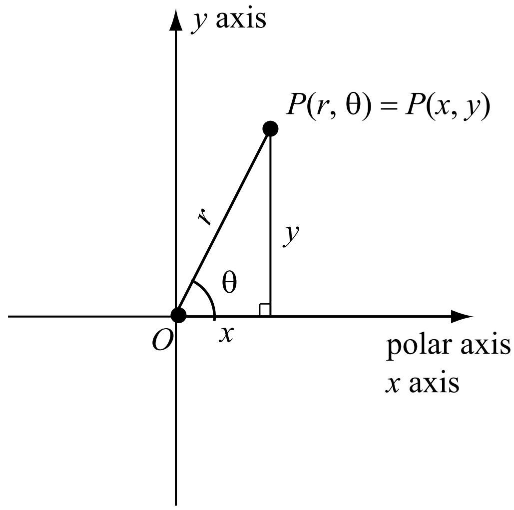

Figure 5.8: Polar and Rectangular Coordinates

coordinates (r, θ):

$$\cos(\theta) = \frac{x}{r} \qquad\qquad x = r\cos(\theta)$$
$$\sin(\theta) = \frac{y}{r} \qquad\qquad y = r\sin(\theta)$$
$$\tan(\theta) = \frac{y}{x} \text{ if } x \neq 0 \qquad\qquad x^2 + y^2 = r^2$$

Coordinate Conversion

To determine the rectangular coorinates (x, y) of a point whose polar coordinates (r, θ) are known, use the equations

$$x = r\cos(\theta) \qquad\qquad y = r\sin(\theta).$$

To determine the polar coordinates (r, θ) of a point whose rectangular coordinates (x, y) are known, use the equation $r^2 = x^2 + y^2$ to determine r and determine an angle θ so that

$$\tan(\theta) = \frac{y}{x} \text{ if } x \neq 0 \qquad \cos(\theta) = \frac{x}{r} \qquad \sin(\theta) = \frac{y}{r}.$$

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When determining the polar coordinates of a point, we usually choose the positive value for r. We can use an inverse trigonometric function to help determine θ but we must be careful to place θ in the proper quadrant by using the signs of x and y. Note that if $x = 0$, we can use $\theta = \dfrac{\pi}{2}$ or $\theta = \dfrac{3\pi}{2}$.

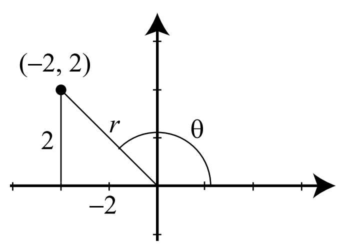
Progress Check 5.18 (Converting from Polar to Rectangular Coordinates)
Determine rectangular coordinates for each of the following points in polar coordinates:

1. $\left(3, \dfrac{\pi}{3}\right)$ **2.** $\left(5, \dfrac{11\pi}{6}\right)$ **3.** $\left(-5, \dfrac{3\pi}{4}\right)$

When we convert from rectangular coordinates to polar coordinates, we must be careful and use the signs of x and y to determine the proper quadrant for the angle θ. In many situations, it might be easier to first determine the reference angle for the angle θ and then use the signs of x and y to determine θ.

Example 5.19 (Converting from Rectangular to Polar Coordinates)
To determine polar coordinates for the the point $(-2, 2)$ in rectangular coordinates, we first draw a picture and note that

$$r^2 = (-2)^2 + 2^2 = 8.$$



Since it is usually easier to work with a positive value for r, we will use $r = \sqrt{8}$.

We also see that $\tan(\theta) = \dfrac{3}{-3} = -1$. We can use many different values for θ but to keep it easy, we use θ as shown in the diagram. For the reference angle $\hat{\theta}$, we have $\tan(\hat{\theta}) = 1$ and so $\hat{\theta} = \dfrac{\pi}{4}$. Since $-2 < 0$ and $2 > 0$, θ is in the second quadrant, and we have

$$\theta = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

So the point $(-2, 2)$ in rectangular coordinates has polar coordinates $\left(\sqrt{8}, \dfrac{3\pi}{4}\right)$.

Progress Check 5.20 (Converting from Rectangular to Polar Coordinates)

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Determine polar coordinates for each of the following points in rectangular coordinates:

1. $\left(6, 6\sqrt{3}\right)$ **2.** $(0, -4)$ **3.** $(-4, 5)$

In each case, use a positive radial distance r and a polar angle θ with $0 \le \theta < 2\pi$. An inverse trigonometric function will need to be used for (3).

The Graph of a Polar Equation

The graph an equation on the rectangular coordinate system consists of all points (x, y) that satisfy the equation. The equation can often be written in the form of a function such as $y = f(x)$. In this case, a point (a, b) is on the graph of this function if and only if $b = f(a)$. In a similar manner,

An equation whose variables are polar coordinates (usually r and θ) is called a **polar equation**. The **graph of a polar equation** is the set of all points whose polar coordinates (r, θ) satisfy the given equation.

An example of a polar equation is $r = 4\sin(\theta)$. For this equation, notice that

- If $\theta = 0$, then $r = 4\sin(0) = 0$ and so the point $(0, 0)$ (in polar coordinates) is on the graph of this equation.
- If $\theta = \dfrac{\pi}{6}$, then $r = 4\sin\left(\dfrac{\pi}{6}\right) = 4 \cdot \dfrac{1}{2} = 2$ and so $\left(2, \dfrac{\pi}{6}\right)$ is on the graph of this equation. (Remember: for polar coordinates, the value of r is the first coordinate.)

The most basic method for drawing the graph of a polar equation is to plot the points that satisfy the polar equation on polar graph paper as shown in Figure 5.7 and then connect the points with a smooth curve.

Progress Check 5.21 (Graphing a Polar Equation)
The following table shows the values of r and θ for points that are on the graph of the polar equation $r = 4\sin(\theta)$.

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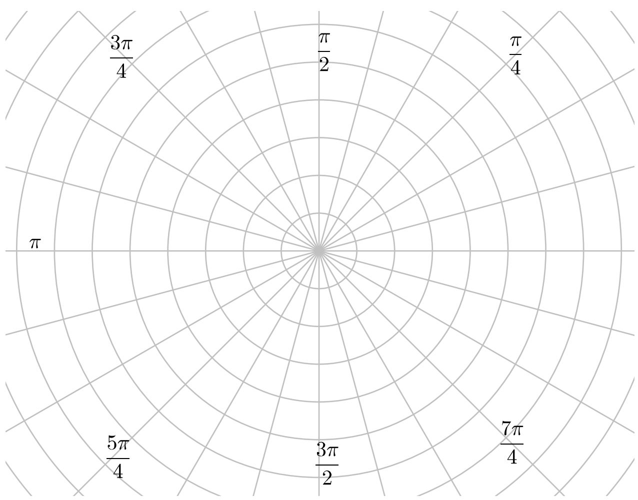
$r = 4\sin(\theta)$	θ
0	0
2	$\dfrac{\pi}{6}$
$2\sqrt{2}$	$\dfrac{\pi}{4}$
$2\sqrt{3}$	$\dfrac{\pi}{3}$
4	$\dfrac{\pi}{2}$

$r = 4\sin(\theta)$	θ
$2\sqrt{3}$	$\dfrac{2\pi}{3}$
$2\sqrt{2}$	$\dfrac{3\pi}{4}$
2	$\dfrac{5\pi}{6}$
0	π

Plot these points on polar graph paper and draw a smooth curve through the points for the graph of the equation $r = 4\sin(\theta)$.



Depending on how carefully we plot the points and how well we draw the curve, the graph in Progress Check 5.21 could be a circle. We can, of course, plot more points. In fact, in Progress Check 5.21, we only used values for θ with $0 \le \theta \le \pi$. The following table shows the values of r and θ for points that are on the graph of the polar equation $r = 4\sin(\theta)$ with $\pi \le \theta \le 2\pi$.

$r = 4\sin(\theta)$	θ
0	π
-2	$\frac{7\pi}{6}$
$-2\sqrt{2}$	$\frac{5\pi}{4}$
$-2\sqrt{3}$	$\frac{4\pi}{3}$
-4	$\frac{3\pi}{2}$

$r = 4\sin(\theta)$	θ
$-2\sqrt{3}$	$\frac{5\pi}{3}$
$-2\sqrt{2}$	$\frac{7\pi}{4}$
-2	$\frac{11\pi}{6}$
0	π

Because of the negative values for r, if we plot these points, we will get the same points we did in Progress Check 5.21. So a good question to ask is, "Do these points really lie on a circle?" We can answer this question by converting the equation $r = 4\sin(\theta)$ into an equivalent equation with rectangular coordinates.

Transforming an Equation from Polar Form to Rectangular Form

The formulas that we used to convert a point in polar coordinates to rectangular coordinates can also be used to convert an equation in polar form to rectangular form. These equations are given in the box on page 326. So let us look at the equation $r = 4\sin(\theta)$ from Progress Check 5.21.

Progress Check 5.22 (Transforming a Polar Equation into Rectangular Form)
We start with the equation $r = 4\sin(\theta)$. We want to transform this into an equation involving x and y. Since $r^2 = x^2 + y^2$, it might be easier to work with r^2 rather than r.

1. Multiply both sides of the equation $r = 4\sin(\theta)$ by r.
2. Now use the equations $r^2 = x^2 + y^2$ and $y = r\sin(\theta)$ to obtain an equivalent equation in x and y.

The graph of the equation the graph of $r = 4\sin(\theta)$ in polar coordinates will be the same as the graph of $x^2 + y^2 = 4y$ in rectangular coordinates. We can now use some algebra from previous mathematics courses to show that this is the graph of a circle. The idea is to collect all terms on the left side of the equation and use completing the square for the terms involving y.

As a reminder, if we have the expression $t^2 + at = 0$, we complete the square by adding $\left(\frac{a}{2}\right)^2$ to both sides of the equation. We will then have a perfect square on the left side of the equation.

$$\begin{aligned} t^2 + at + \left(\frac{a}{2}\right)^2 &= \left(\frac{a}{2}\right)^2 \\ t^2 + at + \frac{a^2}{4} &= \frac{a^2}{4} \\ \left(t + \frac{a}{2}\right)^2 &= \frac{a^2}{4} \end{aligned}$$

So for the equation $x^2 + y^2 = 4y$, we have

$$\begin{aligned} x^2 + y^2 - 4y &= 0 \\ x^2 + y^2 - 4y + 4 &= 4 \\ x^2 + (y-2)^2 &= 2^2 \end{aligned}$$

This is the equation (in rectangular coordinates) of a circle with radius 2 and center at the point $(0, 2)$. We see that this is consistent with the graph we obtained in Progress Check 5.22.

Progress Check 5.23 (Transforming a Polar Equation into Rectangular Form)
Transform the equation $r = 6\cos(\theta)$ into an equation in rectangular coordinates and then explain why the graph of $r = 6\cos(\theta)$ is a circle. What is the radius of this circle and what is its center?

The Polar Grid

We introduced polar graph paper in Figure 5.7. Notice that this consists of concentric circles centered at the pole and lines that pass through the pole. These circles and lines have very simple equations in polar coordinates. For example:

- Consider the equation $r = 3$. In order for a point to be on the graph of this equation, it must lie on a circle of radius 3 whose center is the pole. So the graph of this equation is a circle of radius 3 whose center is the pole. We can also show this by converting the equation $r = 3$ to rectangular form as follows:

$$\begin{aligned} r &= 3 \\ r^2 &= 3^2 \\ x^2 + y^2 &= 9 \end{aligned}$$

In rectangular coordinates, this is the equation of a circle of radius 3 centered at the origin.

- Now consider the equation $\theta = \dfrac{\pi}{4}$. In order for a point to be on the graph of this equation, the line through the pole and this point must make an angle of $\dfrac{\pi}{4}$ radians with the polar axis. If we only allow positive values for r, the graph will be a ray with initial point at the pole that makes an angle of $\dfrac{\pi}{4}$ with the polar axis. However, if we allow r to be any real number, then we obtain the line through the pole that makes an angle of $\dfrac{\pi}{4}$ radians with the polar axis. We can convert this equation to rectangular coordinates as follows:

$$\begin{aligned} \theta &= \frac{\pi}{4} \\ \tan(\theta) &= \tan\left(\frac{\pi}{4}\right) \\ \frac{y}{x} &= 1 \\ y &= x \end{aligned}$$

  This is an equation for a straight line through the origin with a slope of 1.

In general:

The Polar Grid

- If a is a positive real number, then the graph of $r = a$ is a circle of radius a whose center is the pole.
- If b is a real number, then the graph of $\theta = b$ is a line through the pole that makes an angle of b radians with the polar axis.

Concluding Remarks

We have studied just a few graphs of polar equations. There are many interesting graphs that can be generated using polar equations that are very difficult to accomplish in rectangular coordinates. Since the polar coordinate system is based on concentric circles, it should not be surprising that circles with center at the pole would have "simple" equations like $r = a$.

In Progress Checks 5.21 and 5.23, we saw polar equations whose graphs were circles with centers not at the pole. These were special cases of the following:

Polar Equations Whose Graphs Are Circles
If a is a positive real number, then

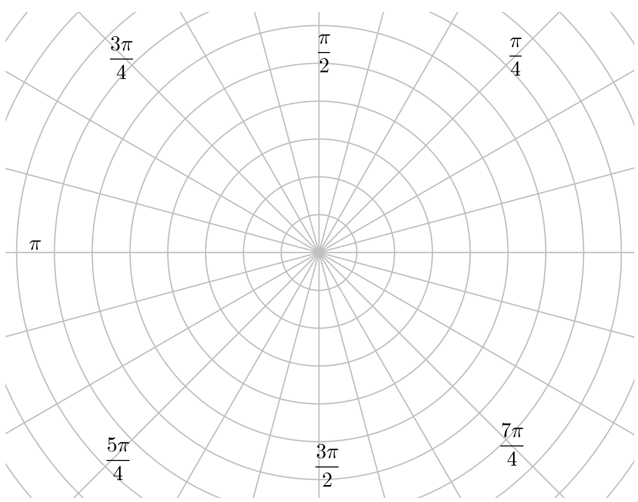
- The graph of $r = 2a\sin(\theta)$ is a circle of radius a with center at the point $(0, a)$ in rectangular coordinates or $\left(a, \dfrac{\pi}{2}\right)$ in polar coordinates.
- The graph of $r = 2a\cos(\theta)$ is a circle of radius a with center at the point $(a, 0)$ in rectangular coordinates or $(a, 0)$ in polar coordinates.

We will explore this and the graphs of other polar equations in the exercises.

Exercises for Section 5.4

⋆ **1.** Plot the following points with the specified polar coordinates.

$$\left(7, \frac{\pi}{6}\right) \qquad \left(3, \frac{3\pi}{4}\right) \qquad \left(2, \frac{-\pi}{3}\right)$$
$$\left(3, \frac{7\pi}{4}\right) \qquad \left(5, -\frac{\pi}{4}\right) \qquad \left(4, \frac{11\pi}{4}\right)$$
$$\left(6, \frac{11\pi}{6}\right) \qquad \left(-3, \frac{2\pi}{3}\right) \qquad \left(-5, \frac{5\pi}{6}\right)$$



2. For each of the following points in polar coordinates, determine three different representations in polar coordinates for the point. Use a positive value

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for the radial distance r for two of the representations and a negative value for the radial distance r for the other representation.

- ⋆ **(a)** $(5, 30°)$
- ⋆ **(b)** $(4, 100°)$
- **(c)** $(-2, 50°)$
- **(d)** $(7, -60°)$

3. For each of the following points in polar coordinates, determine three different representations in polar coordinates for the point. Use a positive value for the radial distance r for two of the representations and a negative value for the radial distance r for the other representation. **Note**: The angles are measured in radians.

- ⋆ **(a)** $\left(5, \dfrac{\pi}{6}\right)$
- ⋆ **(b)** $\left(4, \dfrac{5\pi}{9}\right)$
- **(c)** $\left(-2, \dfrac{5\pi}{18}\right)$
- **(d)** $\left(7, -\dfrac{\pi}{3}\right)$

4. Determine rectangular coordinates for each of the following points in polar coordinates:

- ⋆ **(a)** $\left(10, \dfrac{2\pi}{3}\right)$
- **(b)** $\left(8, \dfrac{7\pi}{6}\right)$
- ⋆ **(c)** $\left(-5, \dfrac{5\pi}{4}\right)$
- **(d)** $\left(10, -\dfrac{2\pi}{3}\right)$
- **(e)** $\left(3, \dfrac{5\pi}{3}\right)$
- **(f)** $\left(6, -\dfrac{\pi}{6}\right)$

5. Determine polar coordinates for each of the following points in rectangular coordinates. Use a positive radial distance r and a polar angle with $0 \le \theta < 2\pi$. When necessary, use an inverse trigonometric function and round the angle (in radians) to the nearest thousandth.

- ⋆ **(a)** $\left(\dfrac{-5\sqrt{3}}{2}, \dfrac{5}{2}\right)$
- ⋆ **(b)** $(3, 5)$
- **(c)** $\left(\sqrt{2}, -\sqrt{2}\right)$
- **(d)** $(-3, -4)$

6. Convert each of the following polar equations into a rectangular equation. If possible, write the rectangular equation with y as a function of x.

⋆ **(a)** $r = 5$

⋆ **(b)** $\theta = \dfrac{\pi}{3}$

(c) $r = 8\cos(\theta)$

⋆ **(d)** $r = 1 - \sin(\theta)$

(e) $r^2 \sin(2\theta) = 1$

(f) $r = 1 - 2\cos(\theta)$

(g) $r = \dfrac{3}{\sin(\theta) + 4\cos(\theta)}$

7. Convert each of the following rectangular equations into a polar equation. If possible, write the polar equation with r as a function of θ.

(a) $x^2 + y^2 = 36$

⋆ **(b)** $y = 4$

(c) $x = 7$

(d) $x^2 - 6x + y^2 = 0$

⋆ **(e)** $x + y = 4$

(f) $y = x^2$

8. Let a be a positive real number.

(a) Convert the polar equation $r = 2a\sin(\theta)$ to rectangular coordinates and then explain why the graph of this equation is a circle. What is the radius of the circle and what is the center of the circle in rectangular coordinates?

(b) Convert the polar equation $r = 2a\cos(\theta)$ to rectangular coordinates and then explain why the graph of this equation is a circle. What is the radius of the circle and what is the center of the circle in rectangular coordinates?