Section 6

Continuous Functions in Metric Spaces

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What does it mean for a function between metric spaces to be continuous at a point?
- What does it mean for a function between metric spaces to be continuous?

Introduction

We have likely had previous experiences with continuous functions. Continuity is an important consideration in optimization problems because a continuous function attains a maximum value and a minimum value on any closed and bounded interval. Continuous functions also satisfy the Intermediate Value Theorem, that a continuous function f takes on all values between $f(a)$ and $f(b)$ on an interval $[a, b]$. An important consequence of the Intermediate Value Theorem is that if f is a continuous function on an interval and $f(a)$ and $f(b)$ have opposite signs, then f must have a root in the interval $[a, b]$. In this section we will begin to explore continuity of functions between metric spaces. Our ultimate goal in future sections is to understand continuous functions well enough that we can define continuity just in terms of open sets.

In calculus we discuss the idea of continuity. A function $f : \mathbb{R} \to \mathbb{R}$ (using the standard Euclidean metric) is continuous at a point a if

$$\lim_{x \to a} f(x) = f(a).$$

This involved providing some explanation about what it means for a function f to have a limit at a point. Intuitively, the idea is that a function f has a limit L at $x = a$ if we can make all of the value of $f(x)$ as close to L as we want by choosing x as close to (but not equal to) a as we need. To extend this informal notion of limit to continuity at a point we would say that a function f is continuous at a point a if if we can make all of the value of $f(x)$ as close to $f(a)$ as we want by choosing x as close to a as we need (now x can equal a).

In order to define continuity in a more general context (in topological spaces) we will need to have a rigorous definition of continuity to work with. We will begin by discussing continuous functions from $\mathbb{R}$ to $\mathbb{R}$, and build from that to continuous functions in metric spaces. These ideas will allow us to ultimately define continuous functions in topological spaces.

We begin by working with continuous functions from $\mathbb{R}$ to $\mathbb{R}$. Our goal is to make more rigorous our informal definition of continuity at a point. To do so will require us to formally defining what we mean by

- making the values of $f(x)$ "as close to $f(a)$ as we want", and
- choosing x "as close to a as we need".

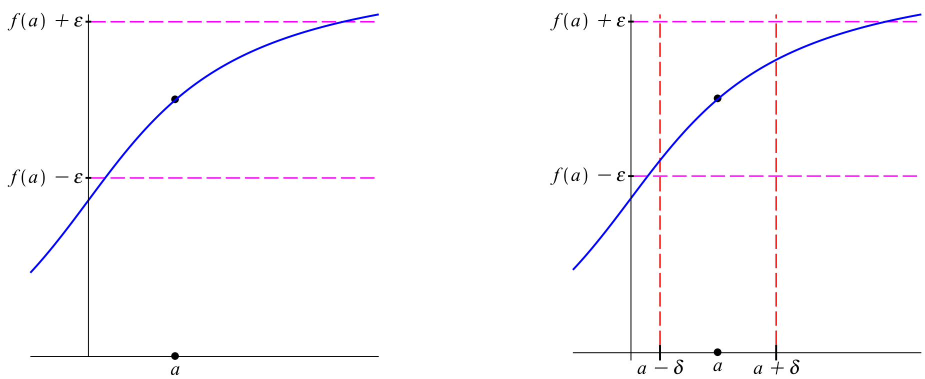

Figure 6.1: Demonstrating the definition of continuity at a point.

Let's deal with the first statement, making the values of $f(x)$ "as close to $f(a)$ as we want". What this means is that if we set any tolerance, say 0.0001, then we can make the values of $f(x)$ within 0.0001 of $f(a)$. Since the absolute value $|f(x) - f(a)|$ measures how close $f(x)$ is to $f(a)$, we can rewrite the statement that the values of $f(x)$ are within 0.0001 of $f(a)$ as $|f(x) - f(a)| < 0.0001$. Of course, 0.0001 may not be as close as we want to $f(a)$, so we need a way to indicate that we can make the values of $f(x)$ arbitrarily close to $f(a)$ – within any tolerance at all. We do this by making the tolerance a parameter, ϵ. Then our job is to make the values of $f(x)$ within ϵ of $f(a)$ regardless of the size of ϵ. We write this as

$$|f(x) - f(a)| < \epsilon.$$

We can picture this as shown at left in Figure 6.1. Here we want to make the values of $f(x)$ lie within an ϵ band of $f(a)$ above and below $f(a)$. That is, we want to be able to make the values of $f(x)$ lie between $f(a) - \epsilon$ and $f(a) + \epsilon$.

Now we have to address the question of how we "make" the values of $f(x)$ to be within ϵ of $f(a)$. Since the values $f(x)$ are the dependent values, dependent on x, we "make" the values of $f(x)$ have the property we want by choosing the inputs x appropriately. In order for f to be continuous at $x = a$, we must be able to find x values close enough to a to force $|f(x) - f(a)| < \epsilon$.

Pictorially, we can see how this might happen in the image at right in Figure 6.1. We need to be able to find an interval around $x = a$ so that the graph of $f(x)$ lies in the ϵ band around $f(a)$ for values of x in that interval. In other words, we need to be able to find some positive number δ so that if x is in the interval $(a - \delta, a + \delta)$, then the graph of $f(x)$ lies in the ϵ band around $y = f(a)$. More formally, if we are given any positive tolerance ϵ, we must be able to find a positive number δ so that if $|x - a| < \delta$ (that is, x is in the interval $(a - \delta, a + \delta)$), then $|f(x) - f(a)| < \epsilon$ (or $f(x)$ lies in the ϵ band around $y = f(a)$).

This gives us a rigorous definition of what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be continuous at a point.

Definition 6.1. A function $f : \mathbb{R} \to \mathbb{R}$ is **continuous at a point** a if, given any $\epsilon > 0$, there exists a $\delta > 0$ so that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$.

Note that the value of δ can depend on the value of a and on ϵ, but not on values of x.

Preview Activity 6.1. The GeoGebra file at `https://www.geogebra.org/m/rym36sqs` will allow us to play around with this definition. Use this GeoGebra applet for the first two problems in this activity.

(1) Enter $f(x) = x\sin(x)$ as your function. (You can change the viewing window coordinates, the base point a, and the function using the input boxes at the left on the screen.) Determine a value of δ so that $|f(x) - f(1)| < 0.5$ whenever $|x - 1| < \delta$. Explain your method.

(2) Now find a value of δ so that $|f(x) - f(2.5)| < 0.25$ whenever $|x - 2.5| < \delta$. Explain your method.

(3)

(a) What is the negation of the definition of continuity at a point? In other words, what do we need to do to show that a function $f : \mathbb{R} \to \mathbb{R}$ is not continuous at a point $x = a$?

(b) Use the negation of the definition to explain why the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} -1 & \text{if } x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$$

is not continuous at $x = 1$.

Continuous Functions Between Metric Spaces

In our preview activity we saw how to formally define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be continuous at a point.

Note that Definition 6.1 depends only on being able to measure how close points are to each other. Since that is precisely what a metric does, we can extend this notion of continuity to define continuity for functions between metric spaces. Continuity is an important idea in topology, and we will work with this idea extensively throughout the semester.

If we let $d_E : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $d_R(x, y) = |x - y|$, then we have seen that d_E is a metric on $\mathbb{R}$ (note that d_E is the Euclidean metric on $\mathbb{R}$). Using this metric we can reformulate what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be continuous at a point.

Definition 6.2 (Alternate Definition)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is **continuous at a point** a if, given any $\epsilon > 0$, there exists a $\delta > 0$ so that $d_E(x, a) < \delta$ implies $d_E(f(x), f(a)) < \epsilon$.

This alternate definition depends on the metric d_E. We could easily replace the metric d with any other metric we choose. This allows us to define continuity at a point for functions between metric spaces.

Definition 6.3. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is **continuous** at $a \in X$ if, given any $\epsilon > 0$, there exists a $\delta > 0$ so that $d_X(x, a) < \delta$ implies $d_Y(f(x), f(a)) < \epsilon$.

Once we have defined continuity at a point, we can define continuous functions.

Definition 6.4. Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is **continuous** if f is continuous at every point in X.

Example 6.5. In general, to prove that a function $f : X \to Y$ is continuous, where (X, d_X) and (Y, d_Y) are metric spaces, we begin by choosing an arbitrary element a in X. Then we let ϵ be a number greater than 0 and show that there is a $\delta > 0$ so that $d_Y(f(x), f(a)) < \epsilon$ whenever $d_X(x, a) < \delta$. The δ we need cannot depend on x (since x isn't known), but can depend on the value of a that we choose, and will likely depend on ϵ as well. That is, there is a function C of only the independent variables a and ϵ that produces the δ, or $\delta = C(a, \epsilon)$. As an example, let $X = \mathbb{R}$ and let d_X be defined as

$$d_X(x, y) = \min\{|x - y|, 1\}.$$

The proof that d_X is a metric is left for Exercise 9. Consider $f : X \to Y$ defined by $f(x) = x^2$, where $(Y, d_Y) = (\mathbb{R}, d_E)$. To show that f is continuous, we let $a \in \mathbb{R}$ and let $\epsilon > 0$.

Scratch work. What happens next is not part of the proof, but shows how we go about finding a δ we need. We are looking for $\delta > 0$ such that $d_X(x, a) < \delta$ implies that $d_E(f(x), f(a)) < \epsilon$. That is, we want to make

$$d_E(f(x), f(a)) = \sqrt{(f(x) - f(a))^2} = |f(x) - f(a)| = |x^2 - a^2| < \epsilon$$

whenever

$$d_X(x, a) = \min\{|x - a|, 1\} < \delta.$$

Now $|x^2 - a^2| = |(x - a)(x + a)| = |x - a|\,|x + a|$. If $d_X(x, a) < \delta$, then $\min\{|x - a|, 1\} < \delta$. If we choose $\delta < 1$, then $d_X(x, a) < \delta < 1$ implies that $|x - a| < 1$ and so $d_X(x, a) = |x - a|$. Now

$$|x + a| = |(x - a) + 2a| \leq |x - a| + 2|a| < 1 + 2|a|.$$

It follows that

$$|x - a|\,|x + a| < \delta(1 + 2|a|).$$

To make this product less that ϵ, we can choose δ such that $\delta(1+2|a|) < \epsilon$ or $\delta < \frac{\epsilon}{1+2|a|}$. That is, there is a function C of ϵ that gives us the δ we want, namely $\delta = C(a, \epsilon) = \left\{1, \frac{\epsilon}{1+2|a|}\right\}$.

Now we ignore this paragraph and present the proof, which is essentially reversing the steps we just made. If the steps can't be reversed, then we have to rethink our argument. The next step in the proof might seem like magic to the uninitiated reader, but we have seen behind the curtain so it isn't a mystery to us.

Let δ be a positive number less than $\min\left\{1, \frac{\epsilon}{1+2|a|}\right\}$. Then

$$d_X(x, a) = \min\{|x-a|, 1\} < \delta$$

implies that $d_X(x, a) < \delta < 1$ and so $d_X(x, a) = |x-a| < \delta < 1$. Then

$$|x+a| = |(x-a)+2a| \leq |x-a| + 2|a| < 1 + 2|a|.$$

It follows that

$$\begin{aligned} d_E(f(x), f(a)) &= \sqrt{(f(x)-f(a))^2} \\ &= |f(x)-f(a)| \\ &= |x^2-a^2| \\ &= |(x-a)(x+a)| \\ &= |x-a|\,|x+a| \\ &< \delta(1+2|a|) \\ &< \left(\frac{\epsilon}{1+2|a|}\right)(1+2|a|) \\ &= \epsilon. \end{aligned}$$

We conclude that f is continuous at every point in X and so f is a continuous function.

Not all functions are continuous as we see in the next example.

Example 6.6. Let $X = Y = \mathbb{R}$ and define $f : X \to Y$ by $f(x) = x$. Let d_X be the Euclidean metric and d_Y the discrete metric. (Recall that $d_Y(x, y) = 1$ whenever $x \neq y$.) Let $a \in X$ and let $0 < \epsilon < 1$.

Let $\delta > 0$, and let $x = a + \frac{\delta}{2}$. Then $x \neq a$ and $d_X(x, a) < \delta$. However,

$$d_Y(f(x), f(a)) = d_Y(x, a) = 1.$$

So if $0 < \epsilon < 1$, there is no $\delta > 0$ such that $d_X(x, a) < \delta$ implies that $d_Y(f(x), f(a)) < \epsilon$. We conclude that f is continuous at no point in X.

Certain functions are always continuous, as the next activity shows.

Activity 6.1.

(a) Let (X, d_X) and (Y, d_Y) be metric spaces, and let $b \in Y$. Define $f : X \to Y$ by $f(x) = b$ for every $x \in X$. Show that f is a continuous function.

(b) Let (X, d) be a metric space. Define the function $i_X : X \to X$ by $i_X(x) = x$ for every $x \in X$. Show that i_X is a continuous function. (The function i_X is called the *identity function* on X.)

(c) Why doesn't the argument in part (b) contradict Example 6.6?

More complicated examples are in the next activity.

Activity 6.2. Let $X = (\mathbb{R}^2, d_T)$ and $Y = (\mathbb{R}^2, d_M)$, where

$$d_T((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

is the taxicab metric and

$$d_M((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}$$

is the max metric. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f((a, b)) = (a + b, b)$.

(a) Is f a continuous function from X to Y? Justify your answer.

(b) Is f a continuous function from Y to X? Justify your answer.

Composites of Continuous Functions

Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces, and suppose $f : X \to Y$ and $g : Y \to Z$ are continuous functions. It seems natural to ask if the composite $g \circ f : X \to Z$ is a continuous function.

Activity 6.3. Let (X, d_X), (Y, d_Y), and (Z, d_Z) be metric spaces, and suppose $f : X \to Y$ and $g : Y \to Z$ are continuous functions. We will prove that $g \circ f$ is a continuous function.

(a) What do we have to do to show that $g \circ f$ is a continuous function? What are the first two steps in our proof?

(b) Let $a \in X$ and let $b = f(a)$. Suppose $\epsilon > 0$ is given. Explain why there must exist a $\delta_1 > 0$ so that $d_Y(y, b) < \delta_1$ implies $d_Z(g(y), g(b)) < \epsilon$.

(c) Now explain why there exists a $\delta_2 > 0$ so that $d_X(x, a) < \delta_2$ implies that $d_Y(f(x), f(a)) < \delta_1$.

(d) Prove that $g \circ f : X \to Z$ is a continuous function.

Continuity is an important concept in topology. We have seen how to define continuity in metric spaces, and we will soon expand on this idea to define continuity without reference to metrics at all. This will allow us to later define continuous functions between arbitrary topological spaces.

Summary

Important ideas that we discussed in this section include the following.

- Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous at $a \in X$ if, given any $\epsilon > 0$, there exists a $\delta > 0$ so that $d_X(x, a) < \delta$ implies $d_Y(f(x), f(a)) < \epsilon$.

- Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous if f is continuous at every point in X.

Exercises

(1) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = |x|$, with the Euclidean metric on both the domain and the codomain. Is f continuous at $x = 0$? Prove your answer.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$ Is f continuous at $x = 0$? Prove your answer.

(3) Let $(Y, d_Y) = (\mathbb{R}, d_E)$, where d_E is the Euclidean metric.

(a) Let $(X, d_X) = (\mathbb{R}^2, d_E)$. Prove or disprove: the function $f : X \to Y$ defined by $f((x_1, x_2)) = x_1 + x_2$ is continuous.

(b) Let $(X, d_X) = (\mathbb{R}^2, d_M)$ where d_M is the max metric. Prove or disprove: the function $f : X \to Y$ defined by $f((x_1, x_2)) = x_1 + x_2$ is continuous.

(4) Let X be any set and define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Exercise 1 on page 39 asks us to show that d is a metric (the discrete metric) on $\mathbb{R}$.

Let (X, d_X) and (Y, d_Y) be metric spaces with d_X the discrete metric. Determine all of the continuous functions f from X to Y.

(5) Let f and g be continuous functions from $(\mathbb{R}, d_E)$ to $(\mathbb{R}, d_E)$.

(a) Let $k \in \mathbb{R}$ with $k \neq 0$ and define $kf : \mathbb{R} \to \mathbb{R}$ by $(kf)(x) = kf(x)$ for all $x \in \mathbb{R}$. Show that kf is a continuous function.

(b) Define $f + g : \mathbb{R} \to \mathbb{R}$ by $(f + g)(x) = f(x) + g(x)$ for all $x \in \mathbb{R}$. Show that $f + g$ is a continuous function.

(6) Let f and g be continuous functions from $(\mathbb{R}, d_E)$ to $(\mathbb{R}, d_E)$. In this exercise we will prove that fg is a continuous function from $\mathbb{R}$ to $\mathbb{R}$. Let a be in $\mathbb{R}$, and follow the steps below to show that fg is continuous at $x = a$. Let ϵ be a positive number.

(a) We will first want to express $f(x)g(x) - f(a)g(a)$ in a more useful way. Use the fact that $f(x) = f(a) + (f(x) - f(a))$ and $g(x) = g(a) + (g(x) - g(a))$ to show that

$$f(x)g(x) - f(a)g(a) = f(a)(g(x) - g(a)) + g(a)(f(x) - f(a)) + (f(x) - f(a))(g(x) - g(a)).$$

(b) Explain why there exist positive numbers δ_1, δ_2, δ_3, and δ_4 such that

$$|f(x) - f(a)| < \sqrt{\frac{\epsilon}{3}} \text{ when } |x - a| < \delta_1$$
$$|g(x) - g(a)| < \sqrt{\frac{\epsilon}{3}} \text{ when } |x - a| < \delta_2$$
$$|f(x) - f(a)| < \frac{\epsilon}{3(1 + |g(a)|)} \text{ when } |x - a| < \delta_3$$
$$|g(x) - g(a)| < \frac{\epsilon}{3(1 + |f(a)|)} \text{ when } |x - a| < \delta_4.$$

(c) Use the results of (a) and (b) to show that fg is continuous at $x = a$. (Hint: $1 + |f(a)| > |f(a)|$.)

(7) Let f and g be functions from $(\mathbb{R}, d_E)$ to $(\mathbb{R}, d_E)$.

(a) Is it true that if $f + g$ is a continuous function, then f and g are continuous functions? Verify your answer.

(b) Is it true that if fg is a continuous function, then f and g are continuous functions? Verify your answer.

(8) Let $f(x) = 2x^2 + 1$ map from $\mathbb{R}$ to $\mathbb{R}$, with both the domain and codomain having the Euclidean metric.

(a) Let $\epsilon = \frac{1}{4}$. Find a value of δ such that $|x - 1| < \delta$ implies that $|f(x) - f(a)| < \epsilon$. You might use the applet at to confirm your value of δ.

(b) Prove that f is continuous at $x = 1$.

(9) Define $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ as

$$d(x, y) = \min\{|x - y|, 1\}.$$

Prove that d is a metric.

(10) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, with both copies of $\mathbb{R}$ having the Euclidean metric. Assume that $f(x) = 0$ whenever x is rational. Prove that $f(x) = 0$ for every $x \in \mathbb{R}$. (Hint: Use Exercise 7 on page 53.)

(11) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$ if x is irrational and $f(x) = 1$ if x is rational. Assume the Euclidean metric on both copies of $\mathbb{R}$. Show that f is not continuous at any point in $\mathbb{R}$. (Hint: Use Exercise 7 on page 53 and Exercise 10 on page 54.)

(12) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 0$ if x is irrational and $g(x) = x$ if x is rational. Assume the Euclidean metric on both copies of $\mathbb{R}$. Show that g is continuous only at 0.

(13) Let X be the set of continuous functions $f : [a, b] \to \mathbb{R}$. Let d^* be the distance function on X defined by

$$d^*(f, g) = \int_a^b |f(t) - g(t)| \, dt,$$

for $f, g \in X$. For each $f \in X$, set

$$I(f) = \int_a^b f(t) \, dt.$$

(a) Determine the value of $d^*(f, g)$ when $f(x) = x^2$, $g(x) = 3-2x$, and $[a, b] = [-3, 3]$.

(b) Determine the value of $I(f)$ if $f(x) = 2x$ and $[a, b] = [0, 2]$.

(c) Prove that the function $I : (X, d^*) \to (\mathbb{R}, d)$ is continuous, where d is the Euclidean metric. (Hint: It helps to start by explicitly writing down what it means for I to be continuous in terms of the metrics d^* and d before trying to prove this statement.)

(14) For each of the following, answer true if the statement is always true. If the statement is only sometimes true or never true, answer false and provide a concrete example to illustrate that the statement is false. If a statement is true, explain why.

(a) Let $f : X \to Y$ be a function, where (X, d_X) and (Y, d_Y) are metric spaces. If d_X is the discrete metric and d_Y is any metric, then f is continuous.

(b) Let $f : X \to Y$ be a function, where (X, d_X) and (Y, d_Y) are metric spaces. If d_Y is the discrete metric and d_X is any metric, then f is continuous.

(c) Let d_1 and d_2 be two metrics on a set X. The identity function $i_X : (X, d_1) \to (X, d_2)$ defined by $i_X(x) = x$ for every $x \in X$ is continuous.

(d) Let f and g be continuous functions from $(\mathbb{R}^2, d_T)$ (the taxicab metric) to $(\mathbb{R}, d_E)$. Then the function $f+g$ from $(\mathbb{R}^2, d_T)$ to $(\mathbb{R}, d_E)$ defined by $(f+g)(x) = f(x)+g(x)$ for every $x \in \mathbb{R}^2$ is a continuous function.

(e) If (X, d_X) and (Y, d_Y) are metric spaces with $y \in Y$, then the constant function $f : X \to Y$ defined by $f(x) = y$ for every $x \in X$ is a continuous function.