Chapter 6

Functions

6.1 Introduction to Functions

Preview Activity 1 (Functions from Previous Courses)
One of the most important concepts in modern mathematics is that of a **function**. In previous mathematics courses, we have often thought of a function as some sort of input-output rule that assigns exactly one output to each input. So in this context, a **function** can be thought of as a procedure for associating with each element of some set, called the **domain of the function**, exactly one element of another set, called the **codomain of the function**. This procedure can be considered an input-output rule. The function takes the input, which is an element of the domain, and produces an output, which is an element of the codomain. In calculus and precalculus, the inputs and outputs were almost always real numbers. So the notation $f(x) = x^2 \sin x$ means the following:

- f is the name of the function.
- $f(x)$ is a real number. It is the output of the function when the input is the real number x. For example,

$$\begin{aligned} f\left(\frac{\pi}{2}\right) &= \left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}\right) \\ &= \frac{\pi^2}{4} \cdot 1 \\ &= \frac{\pi^2}{4}. \end{aligned}$$

For this function, it is understood that the domain of the function is the set $\mathbb{R}$ of all real numbers. In this situation, we think of the domain as the set of all possible inputs. That is, the domain is the set of all possible real numbers x for which a real number output can be determined.

This is closely related to the equation $y = x^2 \sin x$. With this equation, we frequently think of x as the input and y as the output. In fact, we sometimes write $y = f(x)$. The key to remember is that a function must have exactly one output for each input. When we write an equation such as

$$y = \frac{1}{2}x^3 - 1,$$

we can use this equation to define y as a function of x. This is because when we substitute a real number for x (the input), the equation produces exactly one real number for y (the output). We can give this function a name, such as g, and write

$$y = g(x) = \frac{1}{2}x^3 - 1.$$

However, as written, an equation such as

$$y^2 = x + 3$$

cannot be used to define y as a function of x since there are real numbers that can be substituted for x that will produce more than one possible value of y. For example, if $x = 1$, then $y^2 = 4$, and y could be -2 or 2.

Which of the following equations can be used to define a function with $x \in \mathbb{R}$ as the input and $y \in \mathbb{R}$ as the output?

1. $y = x^2 - 2$

2. $y^2 = x + 3$

3. $y = \frac{1}{2}x^3 - 1$

4. $y = \frac{1}{2}x \sin x$

5. $x^2 + y^2 = 4$

6. $y = 2x - 1$

7. $y = \frac{x}{x-1}$

Preview Activity 2 (Some Other Types of Functions)
The domain and codomain of each of the functions in Preview Activity 1 are the set $\mathbb{R}$ of all real numbers, or some subset of $\mathbb{R}$. In most of these cases, the way in which the function associates elements of the domain with elements of the codomain is

by a rule determined by some mathematical expression. For example, when we say that f is the function such that

$$f(x) = \frac{x}{x-1},$$

then the algebraic rule that determines the output of the function f when the input is x is $\dfrac{x}{x-1}$. In this case, we would say that the domain of f is the set of all real numbers not equal to 1 since division by zero is not defined.

However, the concept of a function is much more general than this. The domain and codomain of a function can be any set, and the way in which a function associates elements of the domain with elements of the codomain can have many different forms. The input-output rule for a function can be a formula, a graph, a table, a random process, or a verbal description. We will explore two different examples in this activity.

1. Let b be the function that assigns to each person his or her birthday (month and day). The domain of the function b is the set of all people and the codomain of b is the set of all days in a leap year (i.e., January 1 through December 31, including February 29).

(**a**) Explain why b really is a function. We will call this the **birthday function**.

(**b**) In 1995, Andrew Wiles became famous for publishing a proof of Fermat's Last Theorem. (See A. D. Aczel, *Fermat's Last Theorem: Unlocking the Secret of an Ancient Mathematical Problem*, Dell Publishing, New York, 1996.) Andrew Wiles's birthday is April 11, 1953. Translate this fact into functional notation using the "birthday function" b. That is, fill in the spaces for the following question marks:

$$b(\,?\,) = ?.$$

(**c**) Is the following statement true or false? Explain.

For each day D of the year, there exists a person x such that $b(x) = D$.

(**d**) Is the following statement true or false? Explain.

For any people x and y, if x and y are different people, then $b(x) \neq b(y)$.

2. Let s be the function that associates with each natural number the sum of its distinct natural number divisors. This is called the **sum of the divisors function**. For example, the natural number divisors of 6 are 1, 2, 3, and 6, and so

$$\begin{aligned} s(6) &= 1 + 2 + 3 + 6 \\ &= 12. \end{aligned}$$

(**a**) Calculate $s(k)$ for each natural number k from 1 through 15.

(**b**) Does there exist a natural number n such that $s(n) = 5$? Justify your conclusion.

(**c**) Is it possible to find two different natural numbers m and n such that $s(m) = s(n)$? Explain.

(**d**) Use your responses in (b) and (c) to determine the truth value of each of the following statements.

i. For each $m \in \mathbb{N}$, there exists a natural number n such that $s(n) = m$.

ii. For all $m, n \in \mathbb{N}$, if $m \neq n$, then $s(m) \neq s(n)$.

The Definition of a Function

The concept of a function is much more general than the idea of a function used in calculus or precalculus. In particular, the domain and codomain do not have to be subsets of $\mathbb{R}$. In addition, the way in which a function associates elements of the domain with elements of the codomain can have many different forms. This input-output rule can be a formula, a graph, a table, a random process, a computer algorithm, or a verbal description. Two such examples were introduced in Preview Activity 2.

For the **birthday function**, the domain would be the set of all people and the codomain would be the set of all days in a leap year. For the **sum of the divisors function**, the domain is the set $\mathbb{N}$ of natural numbers, and the codomain could also be $\mathbb{N}$. In both of these cases, the input-output rule was a verbal description of how to assign an element of the codomain to an element of the domain.

We formally define the concept of a function as follows:

Definition. A **function** from a set A to a set B is a rule that associates with each element x of the set A exactly one element of the set B. A function from A to B is also called a **mapping** from A to B.

Function Notation. When we work with a function, we usually give it a name. The name is often a single letter, such as f or g. If f is a function from the set A to the set B, we will write $f\colon A \to B$. This is simply shorthand notation for the fact that f is a function from the set A to the set B. In this case, we also say that f maps A to B.

> **Definition**. Let $f\colon A \to B$. (This is read, "Let f be a function from A to B.") The set A is called the **domain** of the function f, and we write $A = \text{dom}(f)$. The set B is called the **codomain** of the function f, and we write $B = \text{codom}(f)$.
>
> If $a \in A$, then the element of B that is associated with a is denoted by $f(a)$ and is called the **image of a under f**. If $f(a) = b$, with $b \in B$, then a is called a **preimage of b for f**.

Some Function Terminology with an Example. When we have a function $f\colon A \to B$, we often write $y = f(x)$. In this case, we consider x to be an unspecified object that can be chosen from the set A, and we would say that x is the **independent variable** of the function f and y is the **dependent variable** of the function f.

For a specific example, consider the function $g\colon \mathbb{R} \to \mathbb{R}$, where $g(x)$ is defined by the formula

$$g(x) = x^2 - 2.$$

Note that this is indeed a function since given any input x in the domain, $\mathbb{R}$, there is exactly one output $g(x)$ in the codomain, $\mathbb{R}$. For example,

$$\begin{aligned}
g(-2) &= (-2)^2 - 2 = 2,\\
g(5) &= 5^2 - 2 = 23,\\
g(\sqrt{2}) &= \left(\sqrt{2}\right)^2 - 2 = 0,\\
g(-\sqrt{2}) &= \left(-\sqrt{2}\right)^2 - 2 = 0.
\end{aligned}$$

So we say that the image of -2 under g is 2, the image of 5 under g is 23, and so on.

Notice in this case that the number 0 in the codomain has two preimages, $-\sqrt{2}$ and $\sqrt{2}$. This does not violate the mathematical definition of a function since the definition only states that each input must produce one and only one output. That is, each element of the domain has exactly one image in the codomain.

Nowhere does the definition stipulate that two different inputs must produce different outputs.

Finding the preimages of an element in the codomain can sometimes be difficult. In general, if y is in the codomain, to find its preimages, we need to ask, "For which values of x in the domain will we have $y = g(x)$?" For example, for the function g, to find the preimages of 5, we need to find all x for which $g(x) = 5$. In this case, since $g(x) = x^2 - 2$, we can do this by solving the equation

$$x^2 - 2 = 5.$$

The solutions of this equation are $-\sqrt{7}$ and $\sqrt{7}$. So for the function g, the preimages of 5 are $-\sqrt{7}$ and $\sqrt{7}$. We often use set notation for this and say that the set of preimages of 5 for the function g is $\left\{-\sqrt{7}, \sqrt{7}\right\}$.

Also notice that for this function, not every element in the codomain has a preimage. For example, there is no input x such that $g(x) = -3$. This is true since for all real numbers x, $x^2 \geq 0$ and hence $x^2 - 2 \geq -2$. This means that for all x in $\mathbb{R}$, $g(x) \geq -2$.

Finally, note that we introduced the function g with the sentence, "Consider the function $g\colon \mathbb{R} \to \mathbb{R}$, where $g(x)$ is defined by the formula $g(x) = x^2 - 2$." This is one correct way to do this, but we will frequently shorten this to, "Let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2 - 2$", or "Let $g\colon \mathbb{R} \to \mathbb{R}$, where $g(x) = x^2 - 2$."

Progress Check 6.1 (Images and Preimages)
Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - 5x$ for all $x \in \mathbb{R}$, and let $g\colon \mathbb{Z} \to \mathbb{Z}$ be defined by $g(m) = m^2 - 5m$ for all $m \in \mathbb{Z}$.

1. Determine $f(-3)$ and $f\left(\sqrt{8}\right)$.
2. Determine $g(2)$ and $g(-2)$.
3. Determine the set of all preimages of 6 for the function f.
4. Determine the set of all preimages of 6 for the function g.
5. Determine the set of all preimages of 2 for the function f.
6. Determine the set of all preimages of 2 for the function g.

The Codomain and Range of a Function

Besides the domain and codomain, there is another important set associated with a function. The need for this was illustrated in the example of the function g on page 285. For this function, it was noticed that there are elements in the codomain that have no preimage or, equivalently, there are elements in the codomain that are not the image of any element in the domain. The set we are talking about is the subset of the codomain consisting of all images of the elements of the domain of the function, and it is called the range of the function.

> **Definition**. Let $f: A \to B$. The set $\{f(x) \mid x \in A\}$ is called the **range of the function f** and is denoted by range (f). The range of f is sometimes called the **image of the function f** (or the **image of A under f**).

The range of $f: A \to B$ could equivalently be defined as follows:

$$\text{range}(f) = \{y \in B \mid y = f(x) \text{ for some } x \in A\}.$$

Notice that this means that range$(f) \subseteq$ codom(f) but does not necessarily mean that range$(f) =$ codom(f). Whether we have this set equality or not depends on the function f. More about this will be explored in Section 6.3.

Progress Check 6.2 (Codomain and Range)

1. Let b be the function that assigns to each person his or her birthday (month and day).

   (a) What is the domain of this function?

   (b) What is a codomain for this function?

   (c) In Preview Activity 2, we determined that the following statement is true: For each day D of the year, there exists a person x such that $b(x) = D$. What does this tell us about the range of the function b? Explain.

2. Let s be the function that associates with each natural number the sum of its distinct natural number factors.

   (a) What is the domain of this function?

   (b) What is a codomain for this function?

   (c) In Preview Activity 2, we determined that the following statement is false:

For each $m \in \mathbb{N}$, there exists a natural number n such that $s(n) = m$.

Give an example of a natural number m that shows this statement is false, and explain what this tells us about the range of the function s.

The Graph of a Real Function

We will finish this section with methods to visually communicate information about two specific types of functions. The first is the familiar method of graphing functions that was a major part of some previous mathematics courses. For example, consider the function $g: \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^2 - 2x - 1$.

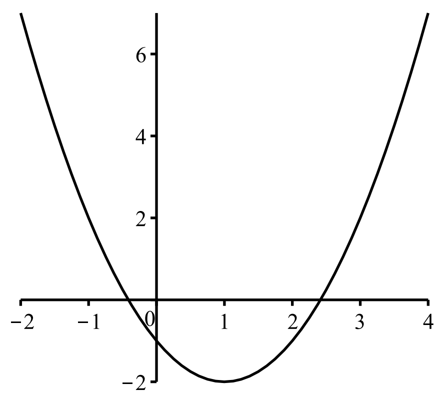


Figure 6.1: Graph of $y = g(x)$, where $g(x) = x^2 - 2x - 1$

Every point on this graph corresponds to an ordered pair (x, y) of real numbers, where $y = g(x) = x^2 - 2x - 1$. Because we use the Cartesian plane when drawing this type of graph, we can only use this type of graph when both the domain and the codomain of the function are subsets of the real numbers $\mathbb{R}$. Such a function is sometimes called a **real function**. The graph of a real function is a visual way to communicate information about the function. For example, the range of g is the set of all y-values that correspond to points on the graph. In this case, the graph of g is a parabola and has a vertex at the point $(1, -2)$. (**Note**: The x-coordinate of the vertex can be found by using calculus and solving the equation $f'(x) = 0$.) Since the graph of the function g is a parabola, we know that the pattern shown on the left end and the right end of the graph continues and we can conclude that the range of g is the set of all $y \in \mathbb{R}$ such that $y \geq -2$. That is,

$$\text{range}(g) = \{y \in \mathbb{R} \mid y \geq -2\}.$$

Progress Check 6.3 (Using the Graph of a Real Function)
The graph in Figure 6.2 shows the graph of (slightly more than) two complete periods for a function $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = A\sin(Bx)$ for some positive real number constants A and B.

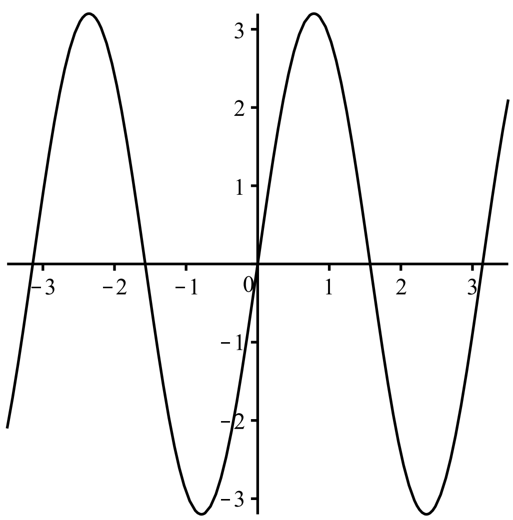


Figure 6.2: Graph of $y = f(x)$

1. We can use the graph to estimate the output for various inputs. This is done by estimating the y-coordinate for the point on the graph with a specified x-coordinate. On the graph, draw vertical lines at $x = -1$ and $x = 2$ and estimate the values of $f(-1)$ and $f(2)$.

2. Similarly, we can estimate inputs of the function that produce a specified output. This is done by estimating the x-coordinates of the points on the graph that have a specified y-coordinate. Draw a horizontal line at $y = 2$ and estimate at least two values of x such that $f(x) = 2$.

3. Use the graph in Figure 6.2 to estimate the range of the function f.

Arrow Diagrams

Sometimes the domain and codomain of a function are small, finite sets. When this is the case, we can define a function simply by specifying the outputs for each input in the domain. For example, if we let $A = \{1, 2, 3\}$ and let $B = \{a, b\}$, we can define a function $F\colon A \to B$ by specifying that

$$F(1) = a,\ F(2) = a, \text{ and } F(3) = b.$$

This is a function since each element of the domain is mapped to exactly one element in B. A convenient way to illustrate or visualize this type of function is with a so-called **arrow diagram** as shown in Figure 6.3. An arrow diagram can

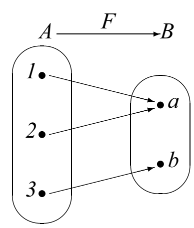


Figure 6.3: Arrow Diagram for a Function

be used when the domain and codomain of the function are finite (and small). We represent the elements of each set with points and then use arrows to show how the elements of the domain are associated with elements of the codomain. For example, the arrow from the point 2 in A to the point a in B represents the fact that $F(2) = a$. In this case, we can use the arrow diagram in Figure 6.3 to conclude that $\text{range}(F) = \{a, b\}$.

Progress Check 6.4 (Working with Arrow Diagrams)
Let $A = \{1, 2, 3, 4\}$ and let $B = \{a, b, c\}$.

1. Which of the arrow diagrams in Figure 6.4 can be used to represent a function from A to B? Explain.

2. For those arrow diagrams that can be used to represent a function from A to B, determine the range of the function.

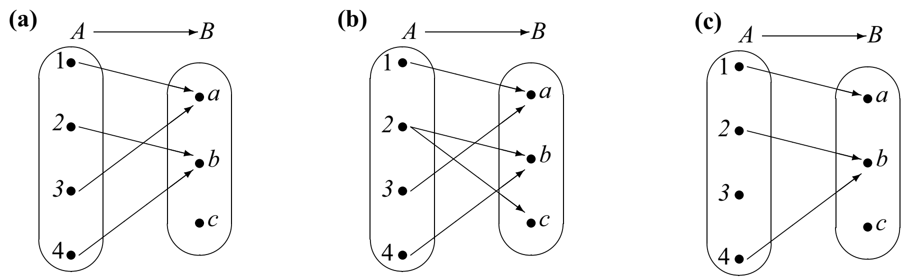


Figure 6.4: Arrow Diagrams

Exercises 6.1

1. Let $f:\mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - 2x$.

- ⋆ **(a)** Evaluate $f(-3)$, $f(-1)$, $f(1)$, and $f(3)$.
- ⋆ **(b)** Determine the set of all of the preimages of 0 and the set of all of the preimages of 4.
- **(c)** Sketch a graph of the function f.
- ⋆ **(d)** Determine the range of the function f.

2. Let $\mathbb{R}^* = \{x \in \mathbb{R} \mid x \geq 0\}$, and let $s:\mathbb{R} \to \mathbb{R}^*$ be defined by $s(x) = x^2$.

- (**a**) Evaluate $s(-3)$, $s(-1)$, $s(1)$, and $s(3)$.
- (**b**) Determine the set of all of the preimages of 0 and the set of all preimages of 2.
- (**c**) Sketch a graph of the function s.
- (**d**) Determine the range of the function s.

3. Let $f:\mathbb{Z} \to \mathbb{Z}$ be defined by $f(m) = 3 - m$.

- ⋆ **(a)** Evaluate $f(-7)$, $f(-3)$, $f(3)$, and $f(7)$.
- ⋆ **(b)** Determine the set of all of the preimages of 5 and the set of all of the preimages of 4.
- ⋆ **(c)** Determine the range of the function f.
- **(d)** This function can be considered a real function since $\mathbb{Z} \subseteq \mathbb{R}$. Sketch a graph of this function. **Note**: The graph will be an infinite set of points that lie on a line. However, it will not be a line since its domain is not $\mathbb{R}$ but is $\mathbb{Z}$.

4. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(m) = 2m + 1$.

(**a**) Evaluate $f(-7)$, $f(-3)$, $f(3)$, and $f(7)$.

* **(b)** Determine the set of all of the preimages of 5 and the set of all of the preimages of 4.

* **(c)** Determine the range of the function f.

* **(d)** Sketch a graph of the function f. See the comments in Exercise (3d).

5. Recall that a **real function** is a function whose domain and codomain are subsets of the real numbers $\mathbb{R}$. (See page 288.) Most of the functions used in calculus are real functions. Quite often, a real function is given by a formula or a graph with no specific reference to the domain or the codomain. In these cases, the usual convention is to assume that the domain of the real function f is the set of all real numbers x for which $f(x)$ is a real number, and that the codomain is $\mathbb{R}$. For example, if we define the (real) function f by

$$f(x) = \frac{x}{x-2},$$

we would be assuming that the domain is the set of all real numbers that are not equal to 2 and that the codomain is $\mathbb{R}$.

Determine the domain and range of each of the following real functions. It might help to use a graphing calculator to plot a graph of the function.

(**a**) The function k defined by $k(x) = \sqrt{x-3}$

* **(b)** The function F defined by $F(x) = \ln(2x - 1)$

(c) The function f defined by $f(x) = 3\sin(2x)$

* **(d)** The function g defined by $g(x) = \dfrac{4}{x^2 - 4}$

(e) The function G defined by $G(x) = 4\cos(\pi x) + 8$

* **6. The number of divisors function**. Let d be the function that associates with each natural number the number of its natural number divisors. That is, $d : \mathbb{N} \to \mathbb{N}$ where $d(n)$ is the number of natural number divisors of n. For example, $d(6) = 4$ since 1, 2, 3, and 6 are the natural number divisors of 6.

(**a**) Calculate $d(k)$ for each natural number k from 1 through 12.

(**b**) Does there exist a natural number n such that $d(n) = 1$? What is the set of preimages of the natural number 1?

(c) Does there exist a natural number n such that $d(n) = 2$? If so, determine the set of all preimages of the natural number 2.

(d) Is the following statement true or false? Justify your conclusion.

For all $m, n \in \mathbb{N}$, if $m \neq n$, then $d(m) \neq d(n)$.

(e) Calculate $d\left(2^k\right)$ for $k = 0$ and for each natural number k from 1 through 6.

(f) Based on your work in Exercise (6e), make a conjecture for a formula for $d(2^n)$ where n is a nonnegative integer. Then explain why your conjecture is correct.

(g) Is the following statement is true or false?

For each $n \in \mathbb{N}$, there exists a natural number m such that $d(m) = n$.

7. In Exercise (6), we introduced the **number of divisors function** d. For this function, $d\colon \mathbb{N} \to \mathbb{N}$, where $d(n)$ is the number of natural number divisors of n.

A function that is related to this function is the so-called **set of divisors function**. This can be defined as a function S that associates with each natural number the set of its distinct natural number factors. For example, $S(6) = \{1, 2, 3, 6\}$ and $S(10) = \{1, 2, 5, 10\}$.

⋆ **(a)** Discuss the function S by carefully stating its domain, codomain, and its rule for determining outputs.

⋆ **(b)** Determine $S(n)$ for at least five different values of n.

⋆ **(c)** Determine $S(n)$ for at least three different prime number values of n.

(d) Does there exist a natural number n such that $\text{card}(S(n)) = 1$? Explain. [Recall that $\text{card}(S(n))$ is the number of elements in the set $S(n)$.]

(e) Does there exist a natural number n such that $\text{card}(S(n)) = 2$? Explain.

(f) Write the output for the function d in terms of the output for the function S. That is, write $d(n)$ in terms of $S(n)$.

(g) Is the following statement true or false? Justify your conclusion.

For all natural numbers m and n, if $m \neq n$, then $S(m) \neq S(n)$.

(h) Is the following statement true or false? Justify your conclusion.

For all sets T that are subsets of $\mathbb{N}$, there exists a natural number n such that $S(n) = T$.

Explorations and Activities

8. Creating Functions with Finite Domains. Let $A = \{a, b, c, d\}$, $B = \{a, b, c\}$, and $C = \{s, t, u, v\}$. In each of the following exercises, draw an arrow diagram to represent your function when it is appropriate.

(**a**) Create a function $f: A \to C$ whose range is the set C or explain why it is not possible to construct such a function.

(**b**) Create a function $f: A \to C$ whose range is the set $\{u, v\}$ or explain why it is not possible to construct such a function.

(**c**) Create a function $f: B \to C$ whose range is the set C or explain why it is not possible to construct such a function.

(**d**) Create a function $f: A \to C$ whose range is the set $\{u\}$ or explain why it is not possible to construct such a function.

(**e**) If possible, create a function $f: A \to C$ that satisfies the following condition:

For all $x, y \in A$, if $x \neq y$, then $f(x) \neq f(y)$.

If it is not possible to create such a function, explain why.

(**f**) If possible, create a function $f: A \to \{s, t, u\}$ that satisfies the following condition:

For all $x, y \in A$, if $x \neq y$, then $f(x) \neq f(y)$.

If it is not possible to create such a function, explain why.

6.2 More about Functions

In Section 6.1, we have seen many examples of functions. We have also seen various ways to represent functions and to convey information about them. For example, we have seen that the rule for determining outputs of a function can be given by a formula, a graph, or a table of values. We have also seen that sometimes it is more convenient to give a verbal description of the rule for a function. In cases where the domain and codomain are small, finite sets, we used an arrow diagram to convey information about how inputs and outputs are associated without explicitly stating a rule. In this section, we will study some types of functions, some of which we may not have encountered in previous mathematics courses.

Preview Activity 1 (**The Number of Diagonals of a Polygon**)
A **polygon** is a closed plane figure formed by the joining of three or more straight

lines. For example, a **triangle** is a polygon that has three sides; a **quadrilateral** is a polygon that has four sides and includes squares, rectangles, and parallelograms; a **pentagon** is a polygon that has five sides; and an **octagon** is a polygon that has eight sides. A **regular polygon** is one that has equal-length sides and congruent interior angles.

A **diagonal of a polygon** is a line segment that connects two nonadjacent vertices of the polygon. In this activity, we will assume that all polygons are **convex polygons** so that, except for the vertices, each diagonal lies inside the polygon. For example, a triangle (3-sided polygon) has no diagonals and a rectangle has two diagonals.

1. How many diagonals does any quadrilateral (4-sided polygon) have?

2. Let $D = \mathbb{N} - \{1, 2\}$. Define $d\colon D \to \mathbb{N} \cup \{0\}$ so that $d(n)$ is the number of diagonals of a convex polygon with n sides. Determine the values of $d(3)$, $d(4)$, $d(5)$, $d(6)$, $d(7)$, and $d(8)$. Arrange the results in the form of a table of values for the function d.

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by
$$f(x) = \frac{x\,(x-3)}{2}.$$
Determine the values of $f(0)$, $f(1)$, $f(2)$, $f(3)$, $f(4)$, $f(5)$, $f(6)$, $f(7)$, $f(8)$, and $f(9)$. Arrange the results in the form of a table of values for the function f.

4. Compare the functions in Parts (2) and (3). What are the similarities between the two functions and what are the differences? Should these two functions be considered equal functions? Explain.

Preview Activity 2 (Derivatives)
In calculus, we learned how to find the derivatives of certain functions. For example, if $f(x) = x^2(\sin x)$, then we can use the product rule to obtain
$$f'(x) = 2x(\sin x) + x^2(\cos x).$$

1. If possible, find the derivative of each of the following functions:

(**a**) $f(x) = x^4 - 5x^3 + 3x - 7$

(**b**) $g(x) = \cos(5x)$

(**c**) $h(x) = \dfrac{\sin x}{x}$

(**d**) $k(x) = e^{-x^2}$

(**e**) $r(x) = |x|$

2. Is it possible to think of differentiation as a function? Explain. If so, what would be the domain of the function, what could be the codomain of the function, and what is the rule for computing the element of the codomain (output) that is associated with a given element of the domain (input)?

Functions Involving Congruences

Theorem 3.31 and Corollary 3.32 (see page 150) state that an integer is congruent (mod n) to its remainder when it is divided by n. (Recall that we always mean the remainder guaranteed by the Division Algorithm, which is the least nonnegative remainder.) Since this remainder is unique and since the only possible remainders for division by n are $0, 1, 2, \ldots, n-1$, we then know that each integer is congruent, modulo n, to precisely one of the integers $0, 1, 2, \ldots, n-1$. So for each natural number n, we will define a new set R_n as the set of remainders upon division by n. So

$$R_n = \{0, 1, 2, \ldots, n-1\}.$$

For example, $R_4 = \{0, 1, 2, 3\}$ and $R_6 = \{0, 1, 2, 3, 4, 5\}$. We will now explore a method to define a function from R_6 to R_6.

For each $x \in R_6$, we can compute $x^2 + 3$ and then determine the value of r in R_6 so that

$$\left(x^2 + 3\right) \equiv r \pmod 6.$$

Since r must be in R_6, we must have $0 \leq r < 6$. The results are shown in the following table.

x	r where $(x^2+3) \equiv r \pmod 6$
0	3
1	4
2	1

x	r where $(x^2+3) \equiv r \pmod 6$
3	0
4	1
5	4

Table 6.1: Table of Values Defined by a Congruence

The value of x in the first column can be thought of as the input for a function with the value of r in the second column as the corresponding output. Each input produces exactly one output. So we could write

$$f : R_6 \to R_6 \text{ by } f(x) = r \text{ where } (x^2 + 3) \equiv r \pmod 6.$$

This description and the notation for the outputs of this function are quite cumbersome. So we will use a more concise notation. We will, instead, write

$$\text{Let } f \colon R_6 \to R_6 \text{ by } f(x) = \left(x^2 + 3\right) \pmod 6.$$

Progress Check 6.5 (Functions Defined by Congruences)
We have $R_5 = \{0, 1, 2, 3, 4\}$. Define

$$f \colon R_5 \to R_5 \text{ by } f(x) = x^4 \pmod 5, \text{ for each } x \in R_5;$$
$$g \colon R_5 \to R_5 \text{ by } g(x) = x^5 \pmod 5, \text{ for each } x \in R_5.$$

1. Determine $f(0)$, $f(1)$, $f(2)$, $f(3)$, and $f(4)$ and represent the function f with an arrow diagram.

2. Determine $g(0)$, $g(1)$, $g(2)$, $g(3)$, and $g(4)$ and represent the function g with an arrow diagram.

Equality of Functions

The idea of equality of functions has been in the background of our discussion of functions, and it is now time to discuss it explicitly. The preliminary work for this discussion was Preview Activity 1, in which $D = \mathbb{N} - \{1, 2\}$, and there were two functions:

- $d \colon D \to \mathbb{N} \cup \{0\}$, where $d(n)$ is the number of diagonals of a convex polygon with n sides
- $f \colon \mathbb{R} \to \mathbb{R}$, where $f(x) = \dfrac{x(x-3)}{2}$, for each real number x.

In Preview Activity 1, we saw that these two functions produced the same outputs for certain values of the input (independent variable). For example, we can verify that

$$\begin{aligned} d(3) = f(3) = 0, & \qquad & d(4) = f(4) = 2, \\ d(5) = f(5) = 5, & \qquad \text{and} \qquad & d(6) = f(6) = 9. \end{aligned}$$

Although the functions produce the same outputs for some inputs, these are two different functions. For example, the outputs of the function f are determined by a formula, and the outputs of the function d are determined by a verbal description. This is not enough, however, to say that these are two different functions. Based on the evidence from Preview Activity 1, we might make the following conjecture:

$$\text{For } n \geq 3, \ d(n) = \frac{n(n-3)}{2}.$$

Although we have not proved this statement, it is a true statement. (See Exercise 6.) However, we know the function d and the function f are not the same function. For example,

- $f(0) = 0$, but 0 is not in the domain of d;
- $f(\pi) = \dfrac{\pi(\pi - 3)}{2}$, but π is not in the domain of d.

We thus see the importance of considering the domain and codomain of each of the two functions in determining whether the two functions are equal or not. This motivates the following definition.

Definition. Two functions f and g are **equal** provided that

- The domain of f equals the domain of g. That is, $\text{dom}(f) = \text{dom}(g)$.
- The codomain of f equals the codomain of g. That is, $\text{codom}(f) = \text{codom}(g)$.
- For each x in the domain of f (which equals the domain of g), $f(x) = g(x)$.

Progress Check 6.6 (Equality of Functions)
Let A be a nonempty set. The **identity function on the set A**, denoted by I_A, is

the function $I_A\colon A \to A$ defined by $I_A(x) = x$ for every x in A. That is, for the identity map, the output is always equal to the input.

For this progress check, we will use the functions f and g from Progress Check 6.5. The identity function on the set R_5 is

$$I_{R_5} : R_5 \to R_5 \text{ by } I_{R_5}(x) = x \pmod 5\text{, for each } x \in R_5.$$

Is the identity function on R_5 equal to either of the functions f or g from Progress Check 6.5? Explain.

Mathematical Processes as Functions

Certain mathematical processes can be thought of as functions. In Preview Activity 2, we reviewed how to find the derivatives of certain functions, and we considered whether or not we could think of this differentiation process as a function. If we use a differentiable function as the input and consider the derivative of that function to be the output, then we have the makings of a function. Computer algebra systems such as *Maple* and *Mathematica* have this derivative function as one of their predefined operators.

Different computer algebra systems will have different syntax for entering functions and for the derivative function. The first step will be to input a real function f. This is usually done by entering a formula for $f(x)$, which is valid for all real numbers x for which $f(x)$ is defined. The next step is to apply the derivative function to the function f. For purposes of illustration, we will use D to represent this derivative function. So this function will give $D(f) = f'$.

For example, if we enter

$$f(x) = x^2 \sin(x)$$

for the function f, we will get

$$D(f) = f'\text{, where } f'(x) = 2x\sin(x) + x^2\cos(x).$$

We must be careful when determining the domain for the derivative function since there are functions that are not differentiable. To make things reasonably easy, we will let F be the set of all real functions that are differentiable and call this the domain of the derivative function D. We will use the set T of all real functions as the codomain. So our function D is

$$D\colon F \to T \text{ by } D(f) = f'.$$

Progress Check 6.7 (Average of a Finite Set of Numbers)
Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite set whose elements are the distinct real numbers $a_1, a_2, \ldots, a_n$. We define the **average of the set** A to be the real number $\bar{A}$, where

$$\bar{A} = \frac{a_1 + a_2 + \cdots + a_n}{n}.$$

1. Find the average of $A = \{3, 7, -1, 5\}$.
2. Find the average of $B = \{7, -2, 3.8, 4.2, 7.1\}$.
3. Find the average of $C = \left\{\sqrt{2}, \sqrt{3}, \pi - \sqrt{3}\right\}$.
4. Now let $\mathcal{F}(\mathbb{R})$ be the set of all nonempty finite subsets of $\mathbb{R}$. That is, a subset A of $\mathbb{R}$ is in $\mathcal{F}(\mathbb{R})$ if and only if A contains only a finite number of elements. Carefully explain how the process of finding the average of a finite subset of $\mathbb{R}$ can be thought of as a function. In doing this, be sure to specify the domain of the function and the codomain of the function.

Sequences as Functions

A sequence can be considered to be an infinite list of objects that are indexed (subscripted) by the natural numbers (or some infinite subset of $\mathbb{N} \cup \{0\}$). Using this idea, we often write a sequence in the following form:

$$a_1, a_2, \ldots, a_n, \ldots.$$

In order to shorten our notation, we will often use the notation $\langle a_n \rangle$ to represent this sequence. Sometimes a formula can be used to represent the terms of a sequence, and we might include this formula as the nth term in the list for a sequence such as in the following example:

$$1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots.$$

In this case, the n^{th} term of the sequence is $\dfrac{1}{n}$. If we know a formula for the n^{th} term, we often use this formula to represent the sequence. For example, we might say

$$\text{Define the sequence } \langle a_n \rangle \text{ by } a_n = \frac{1}{n} \text{ for each } n \in \mathbb{N}.$$

This shows that this sequence is a function with domain $\mathbb{N}$. If it is understood that the domain is $\mathbb{N}$, we could refer to this as the sequence $\left\langle \dfrac{1}{n} \right\rangle$. Given an element of the domain, we can consider a_n to be the output. In this case, we have used subscript notation to indicate the output rather than the usual function notation. We could just as easily write

$$a(n) = \frac{1}{n} \text{ instead of } a_n = \frac{1}{n}.$$

We make the following formal definition.

Definition. An (infinite) **sequence** is a function whose domain is $\mathbb{N}$ or some infinite subset of $\mathbb{N} \cup \{0\}$.

Progress Check 6.8 (Sequences)
Find the sixth and tenth terms of the following sequences, each of whose domain is $\mathbb{N}$:

1. $\dfrac{1}{3}, \dfrac{1}{6}, \dfrac{1}{9}, \dfrac{1}{12}, \ldots$

2. $\langle a_n \rangle$, where $a_n = \dfrac{1}{n^2}$ for each $n \in \mathbb{N}$

3. $\langle (-1)^n \rangle$

Functions of Two Variables

In Section 5.4, we learned how to form the Cartesian product of two sets. Recall that a Cartesian product of two sets is a set of ordered pairs. For example, the set $\mathbb{Z} \times \mathbb{Z}$ is the set of all ordered pairs, where each coordinate of an ordered pair is an integer. Since a Cartesian product is a set, it could be used as the domain or codomain of a function. For example, we could use $\mathbb{Z} \times \mathbb{Z}$ as the domain of a function as follows:

$$\text{Let } f \colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \text{ be defined by } f(m, n) = 2m + n.$$

- Technically, an element of $\mathbb{Z} \times \mathbb{Z}$ is an ordered pair, and so we should write $f((m, n))$ for the output of the function f when the input is the ordered pair (m, n). However, the double parentheses seem unnecessary in this context and there should be no confusion if we write $f(m, n)$ for the output of the function f when the input is (m, n). So, for example, we simply write

$$f(3, 2) = 2 \cdot 3 + 2 = 8, \text{ and}$$
$$f(-4, 5) = 2 \cdot (-4) + 5 = -3.$$

- Since the domain of this function is $\mathbb{Z} \times \mathbb{Z}$ and each element of $\mathbb{Z} \times \mathbb{Z}$ is an ordered pair of integers, we frequently call this type of function a **function of two variables**.

Finding the preimages of an element of the codomain for the function f, $\mathbb{Z}$, usually involves solving an equation with two variables. For example, to find the preimages of $0 \in \mathbb{Z}$, we need to find all ordered pairs $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ such that $f(m, n) = 0$. This means that we must find all ordered pairs $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ such that

$$2m + n = 0.$$

Three such ordered pairs are $(0, 0)$, $(1, -2)$, and $(-1, 2)$. In fact, whenever we choose an integer value for m, we can find a corresponding integer n such that $2m + n = 0$. This means that 0 has infinitely many preimages, and it may be difficult to specify the set of all of the preimages of 0 using the roster method. One way that can be used to specify this set is to use set builder notation and say that the following set consists of all of the preimages of 0:

$$\{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid 2m + n = 0\} = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid n = -2m\}.$$

The second formulation for this set was obtained by solving the equation $2m + n = 0$ for n.

Progress Check 6.9 (Working with a Function of Two Variables)
Let $g\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $g(m, n) = m^2 - n$ for all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$.

1. Determine $g(0, 3)$, $g(3, -2)$, $g(-3, -2)$, and $g(7, -1)$.
2. Determine the set of all preimages of the integer 0 for the function g. Write your answer using set builder notation.
3. Determine the set of all preimages of the integer 5 for the function g. Write your answer using set builder notation.

Exercises 6.2

⋆ **1.** Let $R_5 = \{0, 1, 2, 3, 4\}$. Define $f: R_5 \to R_5$ by $f(x) = x^2 + 4 \pmod 5$, and define $g: R_5 \to R_5$ by $g(x) = (x+1)(x+4) \pmod 5$.

(**a**) Calculate $f(0)$, $f(1)$, $f(2)$, $f(3)$, and $f(4)$.

(**b**) Calculate $g(0)$, $g(1)$, $g(2)$, $g(3)$, and $g(4)$.

(**c**) Is the function f equal to the function g? Explain.

2. Let $R_6 = \{0, 1, 2, 3, 4, 5\}$. Define $f: R_6 \to R_6$ by $f(x) = x^2 + 4 \pmod 6$, and define $g: R_6 \to R_6$ by $g(x) = (x+1)(x+4) \pmod 6$.

(**a**) Calculate $f(0)$, $f(1)$, $f(2)$, $f(3)$, $f(4)$, and $f(5)$.

(**b**) Calculate $g(0)$, $g(1)$, $g(2)$, $g(3)$, $g(4)$, and $g(5)$.

(**c**) Is the function f equal to the function g? Explain.

⋆ **3.** Let $f: (\mathbb{R} - \{0\}) \to \mathbb{R}$ by $f(x) = \dfrac{x^3 + 5x}{x}$ and let $g: \mathbb{R} \to \mathbb{R}$ by $g(x) = x^2 + 5$.

(**a**) Calculate $f(2)$, $f(-2)$, $f(3)$, and $f(\sqrt{2})$.

(**b**) Calculate $g(0)$, $g(2)$, $g(-2)$, $g(3)$, and $g(\sqrt{2})$.

(**c**) Is the function f equal to the function g? Explain.

(**d**) Now let $h: (\mathbb{R} - \{0\}) \to \mathbb{R}$ by $h(x) = x^2 + 5$. Is the function f equal to the function h? Explain.

4. Represent each of the following sequences as functions. In each case, state a domain, codomain, and rule for determining the outputs of the function. Also, determine if any of the sequences are equal.

⋆ (**a**) $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ldots$

(**b**) $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}, \ldots$

(**c**) $1, -1, 1, -1, 1, -1, \ldots$

⋆ (**d**) $\cos(0), \cos(\pi), \cos(2\pi), \cos(3\pi), \cos(4\pi), \ldots$

5. Let A and B be two nonempty sets. There are two **projection functions** with domain $A \times B$, the Cartesian product of A and B. One projection function will map an ordered pair to its first coordinate, and the other projection function will map the ordered pair to its second coordinate. So we define

$p_1: A \times B \to A$ by $p_1(a, b) = a$ for every $(a, b) \in A \times B$; and

$p_2: A \times B \to B$ by $p_2(a, b) = b$ for every $(a, b) \in A \times B$.

Let $A = \{1, 2\}$ and let $B = \{x, y, z\}$.

⋆ **(a)** Determine the outputs for all possible inputs for the projection function $p_1: A \times B \to A$.

(b) Determine the outputs for all possible inputs for the projection function $p_2: A \times B \to B$.

⋆ **(c)** What is the range of these projection functions?

(d) Is the following statement true or false? Explain.

For all $(m, n), (u, v) \in A \times B$, if $(m, n) \neq (u, v)$, then $p_1(m, n) \neq p_1(u, v)$.

⋆ **6.** Let $D = \mathbb{N} - \{1, 2\}$ and define $d: D \to \mathbb{N} \cup \{0\}$ by $d(n) =$ the number of diagonals of a convex polygon with n sides. In Preview Activity 1, we showed that for values of n from 3 through 8,

$$d(n) = \frac{n(n-3)}{2}.$$

Use mathematical induction to prove that for all $n \in D$,

$$d(n) = \frac{n(n-3)}{2}.$$

Hint: To get an idea of how to handle the inductive step, use a pentagon. First, form all the diagonals that can be made from four of the vertices. Then consider how to make new diagonals when the fifth vertex is used. This may generate an idea of how to proceed from a polygon with k sides to a polygon with $k + 1$ sides.

⋆ **7.** Let $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $f(m, n) = m + 3n$.

(**a**) Calculate $f(-3, 4)$ and $f(-2, -7)$.

(**b**) Determine the set of all the preimages of 4 by using set builder notation to describe the set of all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ such that $f(m, n) = 4$.

8. Let $g\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be defined by $g(m, n) = (2m, m - n)$.

- ⋆ **(a)** Calculate $g(3, 5)$ and $g(-1, 4)$.
- **(b)** Determine all the preimages of $(0, 0)$. That is, find all $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ such that $g(m, n) = (0, 0)$.
- ⋆ **(c)** Determine the set of all the preimages of $(8, -3)$.
- **(d)** Determine the set of all the preimages of $(1, 1)$.
- **(e)** Is the following proposition true or false? Justify your conclusion.

  For each $(s, t) \in \mathbb{Z} \times \mathbb{Z}$, there exists an $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ such that $g(m, n) = (s, t)$.

9. A **2 by 2 matrix over** $\mathbb{R}$ is a rectangular array of four real numbers arranged in two rows and two columns. We usually write this array inside brackets (or parentheses) as follows:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

where a, b, c, and d are real numbers. The **determinant** of the 2 by 2 matrix A, denoted by $\det(A)$, is defined as

$$\det(A) = ad - bc.$$

- ⋆ **(a)** Calculate the determinant of each of the following matrices:

$$\begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}, \text{ and } \begin{bmatrix} 3 & -2 \\ 5 & 0 \end{bmatrix}.$$

- **(b)** Let $\mathcal{M}_2(\mathbb{R})$ be the set of all 2 by 2 matrices over $\mathbb{R}$. The mathematical process of finding the determinant of a 2 by 2 matrix over $\mathbb{R}$ can be thought of as a function. Explain carefully how to do so, including a clear statement of the domain and codomain of this function.

10. Using the notation from Exercise (9), let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be a 2 by 2 matrix over $\mathbb{R}$. The **transpose of the matrix A**, denoted by A^T, is the 2 by 2 matrix over $\mathbb{R}$ defined by

$$A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

(**a**) Calculate the transpose of each of the following matrices:

$$\begin{bmatrix} 3 & 5 \\ 4 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 7 \end{bmatrix}, \text{ and } \begin{bmatrix} 3 & -2 \\ 5 & 0 \end{bmatrix}.$$

(**b**) Let $\mathcal{M}_2(\mathbb{R})$ be the set of all 2 by 2 matrices over $\mathbb{R}$. The mathematical process of finding the transpose of a 2 by 2 matrix over $\mathbb{R}$ can be thought of as a function. Carefully explain how to do so, including a clear statement of the domain and codomain of this function.

Explorations and Activities

11. Integration as a Function. In calculus, we learned that if f is real function that is continuous on the closed interval $[a, b]$, then the definite integral $\int_a^b f(x)\,dx$ is a real number. In fact, one form of the **Fundamental Theorem of Calculus** states that

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

where F is any antiderivative of f, that is, where $F' = f$.

(**a**) Let $[a, b]$ be a closed interval of real numbers and let $C[a, b]$ be the set of all real functions that are continuous on $[a, b]$. That is,

$$C[a, b] = \{f : [a, b] \to \mathbb{R} \mid f \text{ is continuous on } [a, b]\}.$$

i. Explain how the definite integral $\int_a^b f(x)\,dx$ can be used to define a function I from $C[a, b]$ to $\mathbb{R}$.

ii. Let $[a, b] = [0, 2]$. Calculate $I(f)$, where $f(x) = x^2 + 1$.

iii. Let $[a, b] = [0, 2]$. Calculate $I(g)$, where $g(x) = \sin(\pi x)$.

In calculus, we also learned how to determine the indefinite integral $\int f(x)\,dx$ of a continuous function f.

(**b**) Let $f(x) = x^2 + 1$ and $g(x) = \cos(2x)$. Determine $\int f(x)\,dx$ and $\int g(x)\,dx$.

(**c**) Let f be a continuous function on the closed interval $[0, 1]$ and let T be the set of all real functions. Can the process of determining the indefinite integral of a continuous function be used to define a function from $C[0, 1]$ to T? Explain.

(**d**) Another form of the Fundamental Theorem of Calculus states that if f is continuous on the interval $[a, b]$ and if

$$g(x) = \int_a^x f(t)\,dt$$

for each x in $[a, b]$, then $g'(x) = f(x)$. That is, g is an antiderivative of f. Explain how this theorem can be used to define a function from $C[a, b]$ to T, where the output of the function is an antiderivative of the input. (Recall that T is the set of all real functions.)

6.3 Injections, Surjections, and Bijections

Functions are frequently used in mathematics to define and describe certain relationships between sets and other mathematical objects. In addition, functions can be used to impose certain mathematical structures on sets. In this section, we will study special types of functions that are used to describe these relationships that are called injections and surjections. Before defining these types of functions, we will revisit what the definition of a function tells us and explore certain functions with finite domains.

Preview Activity 1 (Functions with Finite Domains)
Let A and B be sets. Given a function $f: A \to B$, we know the following:

- For every $x \in A$, $f(x) \in B$. That is, every element of A is an input for the function f. This could also be stated as follows: For each $x \in A$, there exists a $y \in B$ such that $y = f(x)$.
- For a given $x \in A$, there is exactly one $y \in B$ such that $y = f(x)$.

The definition of a function does not require that different inputs produce different outputs. That is, it is possible to have $x_1, x_2 \in A$ with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. The arrow diagram for the function f in Figure 6.5 illustrates such a function.

Also, the definition of a function does not require that the range of the function must equal the codomain. The range is always a subset of the codomain, but these two sets are not required to be equal. That is, if $g: A \to B$, then it is possible to have a $y \in B$ such that $g(x) \neq y$ for all $x \in A$. The arrow diagram for the function g in Figure 6.5 illustrates such a function.

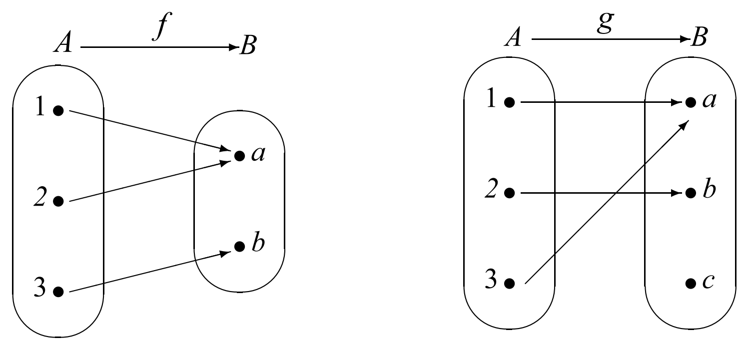


Figure 6.5: Arrow Diagram for Two Functions

Now let $A = \{1, 2, 3\}$, $B = \{a, b, c, d\}$, and $C = \{s, t\}$. Define

$f: A \to B$ by	$g: A \to B$ by	$h: A \to C$ by
$f(1) = a$	$g(1) = a$	$h(1) = s$
$f(2) = b$	$g(2) = b$	$h(2) = t$
$f(3) = c$	$g(3) = a$	$h(3) = s$

1. Which of these functions satisfy the following property for a function F?

   For all $x, y \in \text{dom}(F)$, if $x \neq y$, then $F(x) \neq F(y)$.

2. Which of these functions satisfy the following property for a function F?

   For all $x, y \in \text{dom}(F)$, if $F(x) = F(y)$, then $x = y$.

3. Determine the range of each of these functions.

4. Which of these functions have their range equal to their codomain?

5. Which of the these functions satisfy the following property for a function F?

   For all y in the codomain of F, there exists an $x \in \text{dom}(F)$ such that $F(x) = y$.

Preview Activity 2 (Statements Involving Functions)
Let A and B be nonempty sets and let $f: A \to B$. In Preview Activity 1, we determined whether or not certain functions satisfied some specified properties. These properties were written in the form of statements, and we will now examine these statements in more detail.

1. Consider the following statement:

   For all $x, y \in A$, if $x \neq y$, then $f(x) \neq f(y)$.

   (**a**) Write the contrapositive of this conditional statement.

   (**b**) Write the negation of this conditional statement.

2. Now consider the statement:

   For all $y \in B$, there exists an $x \in A$ such that $f(x) = y$.

   Write the negation of this statement.

3. Let $g : \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 5x + 3$, for all $x \in \mathbb{R}$. Complete the following proofs of the following propositions about the function g.

   Proposition 1. For all $a, b \in \mathbb{R}$, if $g(a) = g(b)$, then $a = b$.

   Proof. We let $a, b \in \mathbb{R}$, and we assume that $g(a) = g(b)$ and will prove that $a = b$. Since $g(a) = g(b)$, we know that

   $$5a + 3 = 5b + 3.$$

   (Now prove that in this situation, $a = b$.)

   Proposition 2. For all $b \in \mathbb{R}$, there exists an $a \in \mathbb{R}$ such that $g(a) = b$.

   Proof. We let $b \in \mathbb{R}$. We will prove that there exists an $a \in \mathbb{R}$ such that $g(a) = b$ by constructing such an a in $\mathbb{R}$. In order for this to happen, we need $g(a) = 5a + 3 = b$.

   (Now solve the equation for a and then show that for this real number a, $g(a) = b$.)

Injections

In previous sections and in Preview Activity 1, we have seen examples of functions for which there exist different inputs that produce the same output. Using more formal notation, this means that there are functions $f : A \to B$ for which there exist $x_1, x_2 \in A$ with $x_1 \neq x_2$ and $f(x_1) = f(x_2)$. The work in the preview activities was intended to motivate the following definition.

Definition. Let $f : A \to B$ be a function from the set A to the set B. The function f is called an **injection** provided that

$$\text{for all } x_1, x_2 \in A, \text{ if } x_1 \neq x_2, \text{ then } f(x_1) \neq f(x_2).$$

When f is an injection, we also say that f is a **one-to-one function**, or that f is an **injective function**.

Notice that the condition that specifies that a function f is an injection is given in the form of a conditional statement. As we shall see, in proofs, it is usually easier to use the contrapositive of this conditional statement. Although we did not define the term then, we have already written the contrapositive for the conditional statement in the definition of an injection in Part (1) of Preview Activity 2. In that activity, we also wrote the negation of the definition of an injection. Following is a summary of this work giving the conditions for f being an injection or not being an injection.

Let $f: A \to B$.

"The function f is an injection" means that

- For all $x_1, x_2 \in A$, if $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$; or
- For all $x_1, x_2 \in A$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$.

"The function f is not an injection" means that

- There exist $x_1, x_2 \in A$ such that $x_1 \neq x_2$ and $f(x_1) = f(x_2)$.

Progress Check 6.10 (Working with the Definition of an Injection)
Now that we have defined what it means for a function to be an injection, we can see that in Part (3) of Preview Activity 2, we proved that the function $g: \mathbb{R} \to \mathbb{R}$ is an injection, where $g(x) = 5x + 3$ for all $x \in \mathbb{R}$. Use the definition (or its negation) to determine whether or not the following functions are injections.

1. $k : A \to B$, where $A = \{a, b, c\}$, $B = \{1, 2, 3, 4\}$, and $k(a) = 4$, $k(b) = 1$, and $k(c) = 3$.

2. $f : A \to C$, where $A = \{a, b, c\}$, $C = \{1, 2, 3\}$, and $f(a) = 2$, $f(b) = 3$, and $f(c) = 2$.

3. $F : \mathbb{Z} \to \mathbb{Z}$ defined by $F(m) = 3m + 2$ for all $m \in \mathbb{Z}$

4. $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) = x^2 - 3x$ for all $x \in \mathbb{R}$

5. $R_5 = \{0, 1, 2, 3, 4\}$ and $s : R_5 \to R_5$ defined by $s(x) = x^3 \pmod 5$ for all $x \in R_5$.

Surjections

In previous sections and in Preview Activity 1, we have seen that there exist functions $f : A \to B$ for which $\text{range}(f) = B$. This means that every element of B is an output of the function f for some input from the set A. Using quantifiers, this means that for every $y \in B$, there exists an $x \in A$ such that $f(x) = y$. One of the objectives of the preview activities was to motivate the following definition.

Definition. Let $f : A \to B$ be a function from the set A to the set B. The function f is called a **surjection** provided that the range of f equals the codomain of f. This means that

for every $y \in B$, there exists an $x \in A$ such that $f(x) = y$.

When f is a surjection, we also say that f is an **onto function** or that f maps ***A* onto *B***. We also say that f is a **surjective function**.

One of the conditions that specifies that a function f is a surjection is given in the form of a universally quantified statement, which is the primary statement used in proving a function is (or is not) a surjection. Although we did not define the term then, we have already written the negation for the statement defining a surjection in Part (2) of Preview Activity 2. We now summarize the conditions for f being a surjection or not being a surjection.

Let $f: A \to B$.

"The function f is a surjection" means that

- $\text{range}(f) = \text{codom}(f) = B$; or
- For every $y \in B$, there exists an $x \in A$ such that $f(x) = y$.

"The function f is not a surjection" means that

- $\text{range}(f) \neq \text{codom}(f)$; or
- There exists a $y \in B$ such that for all $x \in A$, $f(x) \neq y$.

One other important type of function is when a function is both an injection and surjection. This type of function is called a bijection.

Definition. A **bijection** is a function that is both an injection and a surjection. If the function f is a bijection, we also say that f is **one-to-one and onto** and that f is a **bijective function**.

Progress Check 6.11 (Working with the Definition of a Surjection)
Now that we have defined what it means for a function to be a surjection, we can see that in Part (3) of Preview Activity 2, we proved that the function $g : \mathbb{R} \to \mathbb{R}$ is a surjection, where $g(x) = 5x + 3$ for all $x \in \mathbb{R}$. Determine whether or not the following functions are surjections.

1. $k: A \to B$, where $A = \{a, b, c\}$, $B = \{1, 2, 3, 4\}$, and $k(a) = 4$, $k(b) = 1$, and $k(c) = 3$.
2. $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 2$ for all $x \in \mathbb{R}$.
3. $F: \mathbb{Z} \to \mathbb{Z}$ defined by $F(m) = 3m + 2$ for all $m \in \mathbb{Z}$.
4. $s : R_5 \to R_5$ defined by $s(x) = x^3 \pmod 5$ for all $x \in R_5$.

The Importance of the Domain and Codomain

The functions in the next two examples will illustrate why the domain and the codomain of a function are just as important as the rule defining the outputs of a function when we need to determine if the function is a surjection.

Example 6.12 (A Function that Is Neither an Injection nor a Surjection)
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 1$. Notice that

$$f(2) = 5 \text{ and } f(-2) = 5.$$

This is enough to prove that the function f is not an injection since this shows that there exist two different inputs that produce the same output.

Since $f(x) = x^2 + 1$, we know that $f(x) \geq 1$ for all $x \in \mathbb{R}$. This implies that the function f is not a surjection. For example, -2 is in the codomain of f and $f(x) \neq -2$ for all x in the domain of f.

Example 6.13 (A Function that Is Not an Injection but Is a Surjection)
Let $T = \{y \in \mathbb{R} \mid y \geq 1\}$, and define $F : \mathbb{R} \to T$ by $F(x) = x^2 + 1$. As in Example 6.12, the function F is not an injection since $F(2) = F(-2) = 5$.

Is the function F a surjection? That is, does F map $\mathbb{R}$ onto T? As in Example 6.12, we do know that $F(x) \geq 1$ for all $x \in \mathbb{R}$.

To see if it is a surjection, we must determine if it is true that for every $y \in T$, there exists an $x \in \mathbb{R}$ such that $F(x) = y$. So we choose $y \in T$. The goal is to determine if there exists an $x \in \mathbb{R}$ such that

$$\begin{aligned} F(x) &= y, \text{ or} \\ x^2 + 1 &= y. \end{aligned}$$

One way to proceed is to work backward and solve the last equation (if possible) for x. Doing so, we get

$$\begin{aligned} x^2 &= y - 1 \\ x = \sqrt{y-1} \text{ or } x &= -\sqrt{y-1}. \end{aligned}$$

Now, since $y \in T$, we know that $y \geq 1$ and hence that $y - 1 \geq 0$. This means that $\sqrt{y-1} \in \mathbb{R}$. Hence, if we use $x = \sqrt{y-1}$, then $x \in \mathbb{R}$, and

$$\begin{aligned} F(x) &= F\left(\sqrt{y-1}\right) \\ &= \left(\sqrt{y-1}\right)^2 + 1 \\ &= (y-1) + 1 \\ &= y. \end{aligned}$$

This proves that F is a surjection since we have shown that for all $y \in T$, there exists an $x \in \mathbb{R}$ such that $F(x) = y$. Notice that for each $y \in T$, this was a constructive proof of the existence of an $x \in \mathbb{R}$ such that $F(x) = y$.

An Important Lesson. In Examples 6.12 and 6.13, the same mathematical formula was used to determine the outputs for the functions. However, one function was not a surjection and the other one was a surjection. This illustrates the important fact that whether a function is surjective depends not only on the formula that defines the output of the function but also on the domain and codomain of the function.

The next example will show that whether or not a function is an injection also depends on the domain of the function.

Example 6.14 (A Function that Is an Injection but Is Not a Surjection)
Let $\mathbb{Z}^* = \{x \in \mathbb{Z} \mid x \geq 0\} = \mathbb{N} \cup \{0\}$. Define $g: \mathbb{Z}^* \to \mathbb{N}$ by $g(x) = x^2 + 1$. (Notice that this is the same formula used in Examples 6.12 and 6.13.) Following is a table of values for some inputs for the function g.

x	$g(x)$	x	$g(x)$
0	1	3	10
1	2	4	17
2	5	5	26

Notice that the codomain is $\mathbb{N}$, and the table of values suggests that some natural numbers are not outputs of this function. So it appears that the function g is not a surjection.

To prove that g is not a surjection, pick an element of $\mathbb{N}$ that does not appear to be in the range. We will use 3, and we will use a proof by contradiction to prove that there is no x in the domain ($\mathbb{Z}^*$) such that $g(x) = 3$. So we assume that there exists an $x \in \mathbb{Z}^*$ with $g(x) = 3$. Then

$$\begin{aligned} x^2 + 1 &= 3 \\ x^2 &= 2 \\ x &= \pm\sqrt{2}. \end{aligned}$$

But this is not possible since $\sqrt{2} \notin \mathbb{Z}^*$. Therefore, there is no $x \in \mathbb{Z}^*$ with $g(x) = 3$. This means that for every $x \in \mathbb{Z}^*$, $g(x) \neq 3$. Therefore, 3 is not in the range of g, and hence g is not a surjection.

The table of values suggests that different inputs produce different outputs, and hence that g is an injection. To prove that g is an injection, assume that $s, t \in \mathbb{Z}^*$

(the domain) with $g(s) = g(t)$. Then

$$\begin{aligned} s^2 + 1 &= t^2 + 1 \\ s^2 &= t^2. \end{aligned}$$

Since $s, t \in \mathbb{Z}^*$, we know that $s \geq 0$ and $t \geq 0$. So the preceding equation implies that $s = t$. Hence, g is an injection.

An Important Lesson. The functions in the three preceding examples all used the same formula to determine the outputs. The functions in Examples 6.12 and 6.13 are not injections but the function in Example 6.14 is an injection. This illustrates the important fact that whether a function is injective not only depends on the formula that defines the output of the function but also on the domain of the function.

Progress Check 6.15 (**The Importance of the Domain and Codomain**)
Let $\mathbb{R}^+ = \{y \in \mathbb{R} \mid y > 0\}$. Define

$f: \mathbb{R} \to \mathbb{R}$ by $f(x) = e^{-x}$, for each $x \in \mathbb{R}$, and

$g: \mathbb{R} \to \mathbb{R}^+$ by $g(x) = e^{-x}$, for each $x \in \mathbb{R}$.

Determine if each of these functions is an injection or a surjection. Justify your conclusions. **Note**: Before writing proofs, it might be helpful to draw the graph of $y = e^{-x}$. A reasonable graph can be obtained using $-3 \leq x \leq 3$ and $-2 \leq y \leq 10$. Please keep in mind that the graph does not prove any conclusion, but may help us arrive at the correct conclusions, which will still need proof.

Working with a Function of Two Variables

It takes time and practice to become efficient at working with the formal definitions of injection and surjection. As we have seen, all parts of a function are important (the domain, the codomain, and the rule for determining outputs). This is especially true for functions of two variables.

For example, we define $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ by

$$f(a, b) = (2a + b, a - b) \text{ for all } (a, b) \in \mathbb{R} \times \mathbb{R}.$$

Notice that both the domain and the codomain of this function are the set $\mathbb{R} \times \mathbb{R}$. Thus, the inputs and the outputs of this function are ordered pairs of real numbers. For example,

$$f(1,1) = (3,0) \quad \text{and} \quad f(-1,2) = (0,-3).$$

To explore whether or not f is an injection, we assume that $(a,b) \in \mathbb{R} \times \mathbb{R}$, $(c,d) \in \mathbb{R} \times \mathbb{R}$, and $f(a,b) = f(c,d)$. This means that

$$(2a + b, a - b) = (2c + d, c - d).$$

Since this equation is an equality of ordered pairs, we see that

$$\begin{aligned} 2a + b &= 2c + d, \text{ and} \\ a - b &= c - d. \end{aligned}$$

By adding the corresponding sides of the two equations in this system, we obtain $3a = 3c$ and hence, $a = c$. Substituting $a = c$ into either equation in the system give us $b = d$. Since $a = c$ and $b = d$, we conclude that

$$(a,b) = (c,d).$$

Hence, we have shown that if $f(a,b) = f(c,d)$, then $(a,b) = (c,d)$. Therefore, f is an injection.

Now, to determine if f is a surjection, we let $(r,s) \in \mathbb{R} \times \mathbb{R}$, where (r,s) is considered to be an arbitrary element of the codomain of the function f. Can we find an ordered pair $(a,b) \in \mathbb{R} \times \mathbb{R}$ such that $f(a,b) = (r,s)$? Working backward, we see that in order to do this, we need

$$(2a + b, a - b) = (r,s).$$

That is, we need

$$2a + b = r \quad \text{and} \quad a - b = s.$$

Solving this system for a and b yields

$$a = \frac{r+s}{3} \text{ and } b = \frac{r-2s}{3}.$$

Since $r, s \in \mathbb{R}$, we can conclude that $a \in \mathbb{R}$ and $b \in \mathbb{R}$ and hence that $(a,b) \in \mathbb{R} \times \mathbb{R}$.

We now need to verify that for these values of a and b, we get $f(a,b) = (r,s)$. So

$$\begin{aligned} f(a,b) &= f\left(\frac{r+s}{3}, \frac{r-2s}{3}\right) \\ &= \left(2\left(\frac{r+s}{3}\right) + \frac{r-2s}{3}, \frac{r+s}{3} - \frac{r-2s}{3}\right) \\ &= \left(\frac{2r+2s+r-2s}{3}, \frac{r+s-r+2s}{3}\right) \\ &= (r,s). \end{aligned}$$

This proves that for all $(r,s) \in \mathbb{R} \times \mathbb{R}$, there exists $(a,b) \in \mathbb{R} \times \mathbb{R}$ such that $f(a,b) = (r,s)$. Hence, the function f is a surjection. Since f is both an injection and a surjection, it is a bijection.

Progress Check 6.16 (A Function of Two Variables)
Let $g\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be defined by $g(x,y) = 2x + y$, for all $(x,y) \in \mathbb{R} \times \mathbb{R}$.

Note: Be careful! One major difference between this function and the previous example is that for the function g, the codomain is $\mathbb{R}$, not $\mathbb{R} \times \mathbb{R}$. It is a good idea to begin by computing several outputs for several inputs (and remember that the inputs are ordered pairs).

1. Notice that the ordered pair $(1,0) \in \mathbb{R} \times \mathbb{R}$. That is, $(1,0)$ is in the domain of g. Also notice that $g(1,0) = 2$. Is it possible to find another ordered pair $(a,b) \in \mathbb{R} \times \mathbb{R}$ such that $g(a,b) = 2$?
2. Let $z \in \mathbb{R}$. Then $(0,z) \in \mathbb{R} \times \mathbb{R}$ and so $(0,z) \in \operatorname{dom}(g)$. Now determine $g(0,z)$.
3. Is the function g an injection? Is the function g a surjection? Justify your conclusions.

Exercises 6.3

1. (**a**) Draw an arrow diagram that represents a function that is an injection but is not a surjection.

   (**b**) Draw an arrow diagram that represents a function that is an injection and is a surjection.

(c) Draw an arrow diagram that represents a function that is not an injection and is not a surjection.

(d) Draw an arrow diagram that represents a function that is not an injection but is a surjection.

(e) Draw an arrow diagram that represents a function that is not a bijection.

2. We know $R_5 = \{0, 1, 2, 3, 4\}$ and $R_6 = \{0, 1, 2, 3, 4, 5\}$. For each of the following functions, determine if the function is an injection and determine if the function is a surjection. Justify all conclusions.

* **(a)** $f\colon R_5 \to R_5$ by $f(x) = x^2 + 4 \pmod 5$, for all $x \in R_5$

(b) $g\colon R_6 \to R_6$ by $g(x) = x^2 + 4 \pmod 6$, for all $x \in R_6$

* **(c)** $F\colon R_5 \to R_5$ by $F(x) = x^3 + 4 \pmod 5$, for all $x \in R_5$

3. For each of the following functions, determine if the function is an injection and determine if the function is a surjection. Justify all conclusions.

* **(a)** $f\colon \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = 3x + 1$, for all $x \in \mathbb{Z}$.

* **(b)** $F\colon \mathbb{Q} \to \mathbb{Q}$ defined by $F(x) = 3x + 1$, for all $x \in \mathbb{Q}$.

(c) $g\colon \mathbb{R} \to \mathbb{R}$ defined by $g(x) = x^3$, for all $x \in \mathbb{R}$.

(d) $G\colon \mathbb{Q} \to \mathbb{Q}$ defined by $G(x) = x^3$, for all $x \in \mathbb{Q}$.

(e) $k\colon \mathbb{R} \to \mathbb{R}$ defined by $k(x) = e^{-x^2}$, for all $x \in \mathbb{R}$.

(f) $K\colon \mathbb{R}^* \to \mathbb{R}$ defined by $K(x) = e^{-x^2}$, for all $x \in \mathbb{R}^*$.
Note: $\mathbb{R}^* = \{x \in \mathbb{R} \mid x \geq 0\}$.

(g) $K_1\colon \mathbb{R}^* \to T$ defined by $K_1(x) = e^{-x^2}$, for all $x \in \mathbb{R}^*$, where $T = \{y \in \mathbb{R} \mid 0 < y \leq 1\}$.

* **(h)** $h\colon \mathbb{R} \to \mathbb{R}$ defined by $h(x) = \dfrac{2x}{x^2 + 4}$, for all $x \in \mathbb{R}$.

(i) $H\colon \{x \in \mathbb{R} \mid x \geq 0\} \to \left\{ y \in \mathbb{R} \,\middle|\, 0 \leq y \leq \dfrac{1}{2} \right\}$ defined by $H(x) = \dfrac{2x}{x^2 + 4}$, for all $x \in \{x \in \mathbb{R} \mid x \geq 0\}$.

4. For each of the following functions, determine if the function is a bijection. Justify all conclusions.

* **(a)** $F\colon \mathbb{R} \to \mathbb{R}$ defined by $F(x) = 5x + 3$, for all $x \in \mathbb{R}$.

* **(b)** $G\colon \mathbb{Z} \to \mathbb{Z}$ defined by $G(x) = 5x + 3$, for all $x \in \mathbb{Z}$.

(c) $f: (\mathbb{R} - \{4\}) \to \mathbb{R}$ defined by $f(x) = \dfrac{3x}{x-4}$, for all $x \in (\mathbb{R} - \{4\})$.

(d) $g: (\mathbb{R} - \{4\}) \to (\mathbb{R} - \{3\})$ defined by $g(x) = \dfrac{3x}{x-4}$, for all $x \in (\mathbb{R} - \{4\})$.

5. Let $s: \mathbb{N} \to \mathbb{N}$, where for each $n \in \mathbb{N}$, $s(n)$ is the sum of the distinct natural number divisors of n. This is the **sum of the divisors function** that was introduced in Preview Activity 2 from Section 6.1. Is s an injection? Is s a surjection? Justify your conclusions.

6. Let $d: \mathbb{N} \to \mathbb{N}$, where $d(n)$ is the number of natural number divisors of n. This is the **number of divisors function** introduced in Exercise (6) from Section 6.1. Is the function d an injection? Is the function d a surjection? Justify your conclusions.

* **7.** In Preview Activity 2 from Section 6.1 , we introduced the **birthday function**. Is the birthday function an injection? Is it a surjection? Justify your conclusions.

8. **(a)** Let $f: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $f(m,n) = 2m + n$. Is the function f an injection? Is the function f a surjection? Justify your conclusions.

(b) Let $g: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be defined by $g(m,n) = 6m + 3n$. Is the function g an injection? Is the function g a surjection? Justify your conclusions.

* **9.** **(a)** Let $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ be defined by $f(x,y) = (2x, x + y)$. Is the function f an injection? Is the function f a surjection? Justify your conclusions.

(b) Let $g: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ be defined by $g(x,y) = (2x, x + y)$. Is the function g an injection? Is the function g a surjection? Justify your conclusions.

10. Let $f: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x,y) = -x^2y + 3y$, for all $(x,y) \in \mathbb{R} \times \mathbb{R}$. Is the function f an injection? Is the function f a surjection? Justify your conclusions.

11. Let $g: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the function defined by $g(x,y) = (x^3 + 2)\sin y$, for all $(x,y) \in \mathbb{R} \times \mathbb{R}$. Is the function g an injection? Is the function g a surjection? Justify your conclusions.

12. Let A be a nonempty set. The **identity function on the set** A, denoted by I_A, is the function $I_A: A \to A$ defined by $I_A(x) = x$ for every x in A. Is I_A an injection? Is I_A a surjection? Justify your conclusions.

13. Let A and B be two nonempty sets. Define

$$p_1 \colon A \times B \to A \text{ by } p_1(a, b) = a$$

for every $(a, b) \in A \times B$. This is the **first projection function** introduced in Exercise (5) in Section 6.2.

(**a**) Is the function p_1 a surjection? Justify your conclusion.

(**b**) If $B = \{b\}$, is the function p_1 an injection? Justify your conclusion.

(**c**) Under what condition(s) is the function p_1 not an injection? Make a conjecture and prove it.

14. Define $f : \mathbb{N} \to \mathbb{Z}$ as follows: For each $n \in \mathbb{N}$,

$$f(n) = \frac{1 + (-1)^n(2n - 1)}{4}.$$

Is the function f an injection? Is the function f a surjection? Justify your conclusions.

Suggestions. Start by calculating several outputs for the function before you attempt to write a proof. In exploring whether or not the function is an injection, it might be a good idea to use cases based on whether the inputs are even or odd. In exploring whether f is a surjection, consider using cases based on whether the output is positive or less than or equal to zero.

15. Let C be the set of all real functions that are continuous on the closed interval $[0, 1]$. Define the function $A \colon C \to \mathbb{R}$ as follows: For each $f \in C$,

$$A(f) = \int_0^1 f(x)\, dx.$$

Is the function A an injection? Is it a surjection? Justify your conclusions.

16. Let $A = \{(m, n) \mid m \in \mathbb{Z}, n \in \mathbb{Z}, \text{ and } n \neq 0\}$. Define $f \colon A \to \mathbb{Q}$ as follows:

For each $(m, n) \in A$, $f(m, n) = \dfrac{m + n}{n}$.

(**a**) Is the function f an injection? Justify your conclusion.

(**b**) Is the function f a surjection? Justify your conclusion.

17. Evaluation of proofs
See the instructions for Exercise (19) on page 100 from Section 3.1.

(a) Proposition. The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $f(x, y) = (2x + y, x - y)$ is an injection.

Proof. For each (a, b) and (c, d) in $\mathbb{R} \times \mathbb{R}$, if $f(a, b) = f(c, d)$, then

$$(2a + b, a - b) = (2c + d, c - d).$$

We will use systems of equations to prove that $a = c$ and $b = d$.

$$\begin{aligned} 2a + b &= 2c + d \\ a - b &= c - d \\ 3a &= 3c \\ a &= c \end{aligned}$$

Since $a = c$, we see that

$$(2c + b, c - b) = (2c + d, c - d).$$

So $b = d$. Therefore, we have proved that the function f is an injection. ■

(b) Proposition. The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}$ defined by $f(x, y) = (2x + y, x - y)$ is a surjection.

Proof. We need to find an ordered pair such that $f(x, y) = (a, b)$ for each (a, b) in $\mathbb{R} \times \mathbb{R}$. That is, we need $(2x + y, x - y) = (a, b)$, or

$$2x + y = a \qquad \text{and} \qquad x - y = b.$$

Treating these two equations as a system of equations and solving for x and y, we find that

$$x = \frac{a + b}{3} \qquad \text{and} \qquad y = \frac{a - 2b}{3}.$$

Hence, x and y are real numbers, $(x, y) \in \mathbb{R} \times \mathbb{R}$, and

$$\begin{aligned}
f(x, y) &= f\left(\frac{a+b}{3}, \frac{a-2b}{3}\right) \\
&= \left(2\left(\frac{a+b}{3}\right) + \frac{a-2b}{3}, \frac{a+b}{3} - \frac{a-2b}{3}\right) \\
&= \left(\frac{2a+2b+a-2b}{3}, \frac{a+b-a+2b}{3}\right) \\
&= \left(\frac{3a}{3}, \frac{3b}{3}\right) \\
&= (a, b).
\end{aligned}$$

Therefore, we have proved that for every $(a, b) \in \mathbb{R} \times \mathbb{R}$, there exists an $(x, y) \in \mathbb{R} \times \mathbb{R}$ such that $f(x, y) = (a, b)$. This proves that the function f is a surjection. ■

Explorations and Activities

18. Piecewise Defined Functions. We often say that a function is a **piecewise defined function** if it has different rules for determining the output for different parts of its domain. For example, we can define a function $f: \mathbb{R} \to \mathbb{R}$ by giving a rule for calculating $f(x)$ when $x \geq 0$ and giving a rule for calculating $f(x)$ when $x < 0$ as follows:

$$f(x) = \begin{cases} x^2 + 1, & \text{if } x \geq 0; \\ x - 1 & \text{if } x < 0. \end{cases}$$

(**a**) Sketch a graph of the function f. Is the function f an injection? Is the function f a surjection? Justify your conclusions.

For each of the following functions, determine if the function is an injection and determine if the function is a surjection. Justify all conclusions.

(b) $g: [0, 1] \to (0, 1)$ by

$$g(x) = \begin{cases} 0.8, & \text{if } x = 0; \\ 0.5x, & \text{if } 0 < x < 1; \\ 0.6 & \text{if } x = 1. \end{cases}$$

(c) $h: \mathbb{Z} \to \{0, 1\}$ by

$$h(x) = \begin{cases} 0, & \text{if } x \text{ is even}; \\ 1, & \text{if } x \text{ is odd}. \end{cases}$$

19. **Functions Whose Domain is $\mathcal{M}_2(\mathbb{R})$.** Let $\mathcal{M}_2(\mathbb{R})$ represent the set of all 2 by 2 matrices over $\mathbb{R}$.

   (a) Define $\det\colon \mathcal{M}_2(\mathbb{R}) \to \mathbb{R}$ by

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

   This is the **determinant function** introduced in Exercise (9) from Section 6.2. Is the determinant function an injection? Is the determinant function a surjection? Justify your conclusions.

   (b) Define $\text{tran}\colon \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ by

$$\text{tran} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}.$$

   This is the **transpose function** introduced in Exercise (10) from Section 6.2. Is the transpose function an injection? Is the transpose function a surjection? Justify your conclusions.

   (c) Define $F\colon \mathcal{M}_2(\mathbb{R}) \to \mathbb{R}$ by

$$F \begin{bmatrix} a & b \\ c & d \end{bmatrix} = a^2 + d^2 - b^2 - c^2.$$

   Is the function F an injection? Is the function F a surjection? Justify your conclusions.

6.4 Composition of Functions

Preview Activity 1 (Constructing a New Function)
Let $A = \{a, b, c, d\}$, $B = \{p, q, r\}$, and $C = \{s, t, u, v\}$. The arrow diagram in Figure 6.6 shows two functions: $f\colon A \to B$ and $g\colon B \to C$. Notice that if $x \in A$, then $f(x) \in B$. Since $f(x) \in B$, we can apply the function g to $f(x)$, and we obtain $g(f(x))$, which is an element of C.

Using this process, determine $g(f(a))$, $g(f(b))$, $g(f(c))$, and $g(f(d))$. Then explain how we can use this information to define a function from A to C.

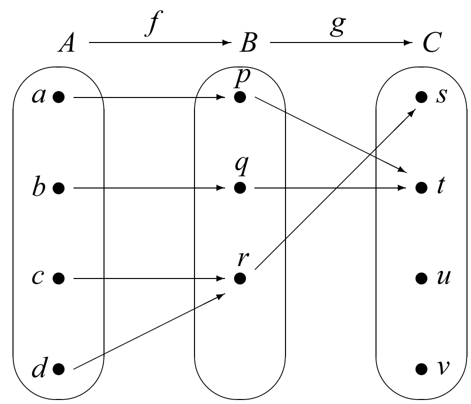


Figure 6.6: Arrow Diagram Showing Two Functions

Preview Activity 2 (Verbal Descriptions of Functions)
The outputs of most real functions we have studied in previous mathematics courses have been determined by mathematical expressions. In many cases, it is possible to use these expressions to give step-by-step verbal descriptions of how to compute the outputs. For example, if

$$f: \mathbb{R} \to \mathbb{R} \text{ is defined by } f(x) = (3x + 2)^3,$$

we could describe how to compute the outputs as follows:

Step	**Verbal Description**	**Symbolic Result**
1	Choose an input.	x
2	Multiply by 3.	$3x$
3	Add 2.	$3x + 2$
4	Cube the result.	$(3x + 2)^3$

Complete step-by-step verbal descriptions for each of the following functions.

1. $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = \sqrt{3x^2 + 2}$, for each $x \in \mathbb{R}$.
2. $g: \mathbb{R} \to \mathbb{R}$ by $g(x) = \sin\left(3x^2 + 2\right)$, for each $x \in \mathbb{R}$.
3. $h: \mathbb{R} \to \mathbb{R}$ by $h(x) = e^{3x^2+2}$, for each $x \in \mathbb{R}$.
4. $G: \mathbb{R} \to \mathbb{R}$ by $G(x) = \ln(x^4 + 3)$, for each $x \in \mathbb{R}$.

5. $k: \mathbb{R} \to \mathbb{R}$ by $k(x) = \sqrt[3]{\dfrac{\sin(4x+3)}{x^2+1}}$, for each $x \in \mathbb{R}$.

Composition of Functions

There are several ways to combine two existing functions to create a new function. For example, in calculus, we learned how to form the product and quotient of two functions and then how to use the product rule to determine the derivative of a product of two functions and the quotient rule to determine the derivative of the quotient of two functions.

The chain rule in calculus was used to determine the derivative of the composition of two functions, and in this section, we will focus only on the composition of two functions. We will then consider some results about the compositions of injections and surjections.

The basic idea of function composition is that when possible, the output of a function f is used as the input of a function g. This can be referred to as "f followed by g" and is called the composition of f and g. In previous mathematics courses, we used this idea to determine a formula for the composition of two real functions.

For example, if

$$f(x) = 3x^2 + 2 \quad \text{and} \quad g(x) = \sin x,$$

then we can compute $g(f(x))$ as follows:

$$\begin{aligned} g(f(x)) &= g\left(3x^2 + 2\right) \\ &= \sin\left(3x^2 + 2\right). \end{aligned}$$

In this case, $f(x)$, the output of the function f, was used as the input for the function g. We now give the formal definition of the composition of two functions.

Definition. Let A, B, and C be nonempty sets, and let $f: A \to B$ and $g: B \to C$ be functions. The **composition of f and g** is the function $g \circ f: A \to C$ defined by

$$(g \circ f)(x) = g\,(f(x))$$

for all $x \in A$. We often refer to the function $g \circ f$ as a **composite function**.

It is helpful to think of the composite function $g \circ f$ as "**f followed by g**." We then refer to f as the **inner function** and g as the **outer function**.

Composition and Arrow Diagrams

The concept of the composition of two functions can be illustrated with arrow diagrams when the domain and codomain of the functions are small, finite sets. Although the term "composition" was not used then, this was done in Preview Activity 1, and another example is given here.

Let $A = \{a, b, c, d\}$, $B = \{p, q, r\}$, and $C = \{s, t, u, v\}$. The arrow diagram in Figure 6.7 shows two functions: $f : A \to B$ and $g : B \to C$.

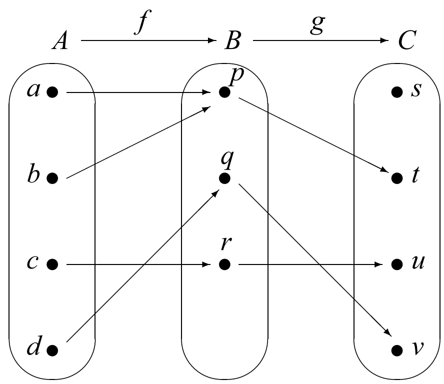


Figure 6.7: Arrow Diagram for Two Functions

If we follow the arrows from the set A to the set C, we will use the outputs of f as inputs of g, and get the arrow diagram from A to C shown in Figure 6.8. This diagram represents the composition of f followed by g.

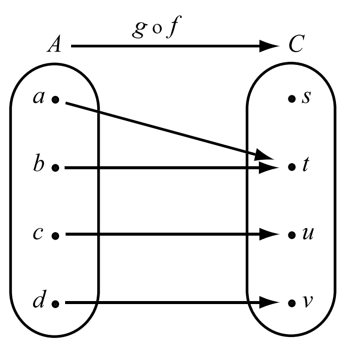


Figure 6.8: Arrow Diagram for $g \circ f : A \to C$

Progress Check 6.17 (The Composition of Two Functions)
Let $A = \{a, b, c, d\}$ and $B = \{1, 2, 3\}$. Define the functions f and g as follows:

$f: A \to B$ defined by $f(a) = 2$, $f(b) = 3$, $f(c) = 1$, and $f(d) = 2$.

$g: B \to B$ defined by $g(1) = 3$, $g(2) = 1$, and $g(3) = 2$.

Create arrow diagrams for the functions f, g, $g \circ f$, and $g \circ g$.

Decomposing Functions

We use the **chain rule** in calculus to find the derivative of a composite function. The first step in the process is to recognize a given function as a composite function. This can be done in many ways, but the work in Preview Activity 2 can be used to decompose a function in a way that works well with the chain rule. The use of the terms "inner function" and "outer function" can also be helpful. The idea is that we use the last step in the process to represent the outer function, and the steps prior to that to represent the inner function. So for the function,

$$f: \mathbb{R} \to \mathbb{R} \text{ by } f(x) = (3x + 2)^3,$$

the last step in the verbal description table was to cube the result. This means that we will use the function g (the cubing function) as the outer function and will use the prior steps as the inner function. We will denote the inner function by h. So we let $h: \mathbb{R} \to \mathbb{R}$ by $h(x) = 3x + 2$ and $g: \mathbb{R} \to \mathbb{R}$ by $g(x) = x^3$. Then

$$\begin{aligned}(g \circ h)(x) &= g\,(h(x)) \\ &= g(3x + 2) \\ &= (3x + 2)^3 \\ &= f(x).\end{aligned}$$

We see that $g \circ h = f$ and, hence, we have "decomposed" the function f. It should be noted that there are other ways to write the function f as a composition of two functions, but the way just described is the one that works well with the chain rule. In this case, the chain rule gives

$$\begin{aligned}f'(x) &= (g \circ h)'(x) \\ &= g'(h(x))\, h'(x) \\ &= 3(h(x))^2 \cdot 3 \\ &= 9(3x + 2)^2\end{aligned}$$

Progress Check 6.18 (Decomposing Functions)
Write each of the following functions as the composition of two functions.

1. $F: \mathbb{R} \to \mathbb{R}$ by $F(x) = (x^2 + 3)^3$
2. $G: \mathbb{R} \to \mathbb{R}$ by $G(x) = \ln(x^2 + 3)$
3. $f: \mathbb{Z} \to \mathbb{Z}$ by $f(x) = |x^2 - 3|$
4. $g: \mathbb{R} \to \mathbb{R}$ by $g(x) = \cos\left(\dfrac{2x - 3}{x^2 + 1}\right)$

Theorems about Composite Functions

If $f: A \to B$ and $g: B \to C$, then we can form the composite function $g \circ f: A \to C$. In Section 6.3, we learned about injections and surjections. We now explore what type of function $g \circ f$ will be if the functions f and g are injections (or surjections).

Progress Check 6.19 (Compositions of Injections and Surjections)
Although other representations of functions can be used, it will be helpful to use arrow diagrams to represent the functions in this progress check. We will use the following sets:

$$A = \{a, b, c\}, \quad B = \{p, q, r\}, \quad C = \{u, v, w, x\}, \quad \text{and} \quad D = \{u, v\}.$$

1. Draw an arrow diagram for a function $f: A \to B$ that is an injection and an arrow diagram for a function $g: B \to C$ that is an injection. In this case, is the composite function $g \circ f: A \to C$ an injection? Explain.
2. Draw an arrow diagram for a function $f: A \to B$ that is a surjection and an arrow diagram for a function $g: B \to D$ that is a surjection. In this case, is the composite function $g \circ f: A \to D$ a surjection? Explain.
3. Draw an arrow diagram for a function $f: A \to B$ that is a bijection and an arrow diagram for a function $g: B \to A$ that is a bijection. In this case, is the composite function $g \circ f: A \to A$ bijection? Explain.

In Progress Check 6.19, we explored some properties of composite functions related to injections, surjections, and bijections. The following theorem contains results that these explorations were intended to illustrate. Some of the proofs will be included in the exercises.

Theorem 6.20. *Let A, B, and C be nonempty sets and assume that $f\colon A \to B$ and $g\colon B \to C$.*

1. *If f and g are both injections, then $(g \circ f)\colon A \to C$ is an injection.*
2. *If f and g are both surjections, then $(g \circ f)\colon A \to C$ is a surjection.*
3. *If f and g are both bijections, then $(g \circ f)\colon A \to C$ is a bijection.*

The proof of Part (1) is Exercise (6). Part (3) is a direct consequence of the first two parts. We will discuss a process for constructing a proof of Part (2). Using the forward-backward process, we first look at the conclusion of the conditional statement in Part (2). The goal is to prove that $g \circ f$ is a surjection. Since $g \circ f\colon A \to C$, this is equivalent to proving that

For all $c \in C$, there exists an $a \in A$ such that $(g \circ f)(a) = c$.

Since this statement in the backward process uses a universal quantifier, we will use the choose-an-element method and choose an arbitrary element c in the set C. The goal now is to find an $a \in A$ such that $(g \circ f)(a) = c$.

Now we can look at the hypotheses. In particular, we are assuming that both $f\colon A \to B$ and $g\colon B \to C$ are surjections. Since we have chosen $c \in C$, and $g\colon B \to C$ is a surjection, we know that

there exists a $b \in B$ such that $g(b) = c$.

Now, $b \in B$ and $f\colon A \to B$ is a surjection. Hence

there exists an $a \in A$ such that $f(a) = b$.

If we now compute $(g \circ f)(a)$, we will see that

$$(g \circ f)(a) = g\left(f(a)\right) = g(b) = c.$$

We can now write the proof as follows:

Proof of Theorem 6.20, Part (2). Let A, B, and C be nonempty sets and assume that $f: A \to B$ and $g: B \to C$ are both surjections. We will prove that $g \circ f: A \to C$ is a surjection.

Let c be an arbitrary element of C. We will prove there exists an $a \in A$ such that $(g \circ f)(a) = c$. Since $g: B \to C$ is a surjection, we conclude that

$$\text{there exists a } b \in B \text{ such that } g(b) = c.$$

Now, $b \in B$ and $f: A \to B$ is a surjection. Hence

$$\text{there exists an } a \in A \text{ such that } f(a) = b.$$

We now see that

$$\begin{aligned}(g \circ f)(a) &= g\left(f(a)\right) \\ &= g(b) \\ &= c.\end{aligned}$$

We have now shown that for every $c \in C$, there exists an $a \in A$ such that $(g \circ f)(a) = c$, and this proves that $g \circ f$ is a surjection. ■

Theorem 6.20 shows us that if f and g are both special types of functions, then the composition of f followed by g is also that type of function. The next question is, "If the composition of f followed by g is an injection (or surjection), can we make any conclusions about f or g?" A partial answer to this question is provided in Theorem 6.21. This theorem will be investigated and proved in the Explorations and Activities for this section. See Exercise (10).

Theorem 6.21. *Let A, B, and C be nonempty sets and assume that $f: A \to B$ and $g: B \to C$.*

1. *If $g \circ f: A \to C$ is an injection, then $f: A \to B$ is an injection.*
2. *If $g \circ f: A \to C$ is a surjection, then $g: B \to C$ is a surjection.*

Exercises 6.4

1. In our definition of the composition of two functions, f and g, we required that the domain of g be equal to the codomain of f. However, it is sometimes possible to form the composite function $g \circ f$ even though $\text{dom}(g) \neq \text{codom}(f)$. For example, let

$f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 + 1$, and let
$g: \mathbb{R} - \{0\} \to \mathbb{R}$ be defined by $g(x) = \dfrac{1}{x}$.

(a) Is it possible to determine $(g \circ f)(x)$ for all $x \in \mathbb{R}$? Explain.

(b) In general, let $f: A \to T$ and $g: B \to C$. Find a condition on the domain of g (other than $B = T$) that results in a meaningful definition of the composite function $g \circ f: A \to C$.

⋆ **2.** Let $h: \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = 3x + 2$ and $g: \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^3$. Determine formulas for the composite functions $g \circ h$ and $h \circ g$. Is the function $g \circ h$ equal to the function $h \circ g$? Explain. What does this tell you about the operation of composition of functions?

⋆ **3.** Following are formulas for certain real functions. Write each of these real functions as the composition of two functions. That is, decompose each of the functions.

(a) $F(x) = \cos(e^x)$

(b) $G(x) = e^{\cos(x)}$

(c) $H(x) = \dfrac{1}{\sin x}$

(d) $K(x) = \cos\left(e^{-x^2}\right)$

4. The **identity function** on a set S, denoted by I_S, is defined as follows: $I_S: S \to S$ by $I_S(x) = x$ for each $x \in S$. Let $f: A \to B$.

⋆ (a) For each $x \in A$, determine $(f \circ I_A)(x)$ and use this to prove that $f \circ I_A = f$.

(b) Prove that $I_B \circ f = f$.

5. ⋆ (a) Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, let $g: \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = \sin x$, and let $h: \mathbb{R} \to \mathbb{R}$ be defined by $h(x) = \sqrt[3]{x}$.

Determine formulas for $[(h \circ g) \circ f](x)$ and $[h \circ (g \circ f)](x)$.

Does this prove that $(h \circ g) \circ f = h \circ (g \circ f)$ for these particular functions? Explain.

(b) Now let A, B, C, and D be sets and let $f: A \to B$, $g: B \to C$, and $h: C \to D$. Prove that $(h \circ g) \circ f = h \circ (g \circ f)$. That is, prove that function composition is an associative operation.

⋆ **6.** Prove Part (1) of Theorem 6.20.

Let A, B, and C be nonempty sets and let $f: A \to B$ and $g: B \to C$. If f and g are both injections, then $g \circ f$ is an injection.

7. For each of the following, give an example of functions $f: A \to B$ and $g: B \to C$ that satisfy the stated conditions, or explain why no such example exists.

⋆ **(a)** The function f is a surjection, but the function $g \circ f$ is not a surjection.

⋆ **(b)** The function f is an injection, but the function $g \circ f$ is not an injection.

(c) The function g is a surjection, but the function $g \circ f$ is not a surjection.

(d) The function g is an injection, but the function $g \circ f$ is not an injection.

(e) The function f is not a surjection, but the function $g \circ f$ is a surjection.

⋆ **(f)** The function f is not an injection, but the function $g \circ f$ is an injection.

(g) The function g is not a surjection, but the function $g \circ f$ is a surjection.

(h) The function g is not an injection, but the function $g \circ f$ is an injection.

8. Let A be a nonempty set and let $f: A \to A$. For each $n \in \mathbb{N}$, define a function $f^n: A \to A$ recursively as follows: $f^1 = f$ and for each $n \in \mathbb{N}$, $f^{n+1} = f \circ f^n$. For example, $f^2 = f \circ f^1 = f \circ f$ and $f^3 = f \circ f^2 = f \circ (f \circ f)$.

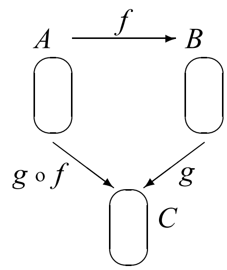
(a) Let $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x + 1$ for each $x \in \mathbb{R}$. For each $n \in \mathbb{N}$ and for each $x \in \mathbb{R}$, determine a formula for $f^n(x)$ and use induction to prove that your formula is correct.

(b) Let $a, b \in \mathbb{R}$ and let $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = ax + b$ for each $x \in \mathbb{R}$. For each $n \in \mathbb{N}$ and for each $x \in \mathbb{R}$, determine a formula for $f^n(x)$ and use induction to prove that your formula is correct.

(c) Now let A be a nonempty set and let $f: A \to A$. Use induction to prove that for each $n \in \mathbb{N}$, $f^{n+1} = f^n \circ f$. (**Note**: You will need to use the result in Exercise (5).)

Explorations and Activities

9. **Exploring Composite Functions**. Let A, B, and C be nonempty sets and let $f: A \to B$ and $g: B \to C$. For this activity, it may be useful to draw your arrow diagrams in a triangular arrangement as follows:



It might be helpful to consider examples where the sets are small. Try constructing examples where the set A has 2 elements, the set B has 3 elements, and the set C has 2 elements.

(**a**) Is it possible to construct an example where $g \circ f$ is an injection, f is an injection, but g is not an injection? Either construct such an example or explain why it is not possible.

(**b**) Is it possible to construct an example where $g \circ f$ is an injection, g is an injection, but f is not an injection? Either construct such an example or explain why it is not possible.

(**c**) Is it possible to construct an example where $g \circ f$ is a surjection, f is a surjection, but g is not a surjection? Either construct such an example or explain why it is not possible.

(**d**) Is it possible to construct an example where $g \circ f$ is surjection, g is a surjection, but f is not a surjection? Either construct such an example or explain why it is not possible.

10. **The Proof of Theorem 6.21**. Use the ideas from Exercise (9) to prove Theorem 6.21. Let A, B, and C be nonempty sets and let $f\colon A \to B$ and $g\colon B \to C$.

(**a**) If $g \circ f\colon A \to C$ is an injection, then $f\colon A \to B$ is an injection.

(**b**) If $g \circ f\colon A \to C$ is a surjection, then $g\colon B \to C$ is a surjection.

Hint: For part (a), start by asking, "What do we have to do to prove that f is an injection?" Start with a similar question for part (b).

6.5 Inverse Functions

For this section, we will use the concept of Cartesian product of two sets A and B, denoted by $A \times B$, which is the set of all ordered pairs (x, y) where $x \in A$ and $y \in B$. That is,

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}.$$

See Preview Activity 2 in Section 5.4 for a more thorough discussion of this concept.

Preview Activity 1 (Functions and Sets of Ordered Pairs)
When we graph a real function, we plot ordered pairs in the Cartesian plane where the first coordinate is the input of the function and the second coordinate is the output of the function. For example, if $g: \mathbb{R} \to \mathbb{R}$, then every point on the graph of g is an ordered pair (x, y) of real numbers where $y = g(x)$. This shows how we can generate ordered pairs from a function. It happens that we can do this with any function. For example, let

$$A = \{1, 2, 3\} \text{ and } B = \{a, b\}.$$

Define the function $F: A \to B$ by

$$F(1) = a, \quad F(2) = b, \quad \text{and} \quad F(3) = b.$$

We can convert each of these to an ordered pair in $A \times B$ by using the input as the first coordinate and the output as the second coordinate. For example, $F(1) = a$ is converted to $(1, a)$, $F(2) = b$ is converted to $(2, b)$, and $F(3) = b$ is converted to $(3, b)$. So we can think of this function as a set of ordered pairs, which is a subset of $A \times B$, and write

$$F = \{(1, a), (2, b), (3, b)\}.$$

Note: Since F is the name of the function, it is customary to use F as the name for the set of ordered pairs.

1. Let $A = \{1, 2, 3\}$ and let $C = \{a, b, c, d\}$. Define the function $g: A \to C$ by $g(1) = a$, $g(2) = b$, and $g(3) = d$. Write the function g as a set of ordered pairs in $A \times C$.

For another example, if we have a real function, such as $g: \mathbb{R} \to \mathbb{R}$ by $g(x) = x^2 - 2$, then we can think of g as the following infinite subset of $\mathbb{R} \times \mathbb{R}$:

$$g = \left\{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = x^2 - 2\right\}.$$

We can also write this as $g = \{(x, x^2 - 2) \mid x \in \mathbb{R}\}$.

2. Let $f: \mathbb{Z} \to \mathbb{Z}$ be defined by $f(m) = 3m + 5$, for all $m \in \mathbb{Z}$. Use set builder notation to write the function f as a set of ordered pairs, and then use the roster method to write the function f as a set of ordered pairs.

So any function $f: A \to B$ can be thought of as a set of ordered pairs that is a subset of $A \times B$. This subset is

$$f = \{(a, f(a)) \mid a \in A\} \quad \text{or} \quad f = \{(a, b) \in A \times B \mid b = f(a)\}.$$

On the other hand, if we started with $A = \{1, 2, 3\}$, $B = \{a, b\}$, and

$$G = \{(1, a), (2, a), (3, b)\} \subseteq A \times B,$$

then we could think of G as a function from A to B with $G(1) = a$, $G(2) = a$, and $G(3) = b$. The idea is to use the first coordinate of each ordered pair as the input, and the second coordinate as the output. However, not every subset of $A \times B$ can be used to define a function from A to B. This is explored in the following questions.

3. Let $f = \{(1, a), (2, a), (3, a), (1, b)\}$. Could this set of ordered pairs be used to define a function from A to B? Explain.

4. Let $g = \{(1, a), (2, b), (3, a)\}$. Could this set of ordered pairs be used to define a function from A to B? Explain.

5. Let $h = \{(1, a), (2, b)\}$. Could this set of ordered pairs be used to define a function from A to B? Explain.

Preview Activity 2 (A Composition of Two Specific Functions)
Let $A = \{a, b, c, d\}$ and let $B = \{p, q, r, s\}$.

1. Construct an example of a function $f: A \to B$ that is a bijection. Draw an arrow diagram for this function.

2. On your arrow diagram, draw an arrow from each element of B back to its corresponding element in A. Explain why this defines a function from B to A.

3. If the name of the function in Part (2) is g, so that $g\colon B \to A$, what are $g(p)$, $g(q)$, $g(r)$, and $g(s)$?

4. Construct a table of values for each of the functions $g \circ f\colon A \to A$ and $f \circ g\colon B \to B$. What do you observe about these tables of values?

The Ordered Pair Representation of a Function

In Preview Activity 1, we observed that if we have a function $f\colon A \to B$, we can generate a set of ordered pairs f that is a subset of $A \times B$ as follows:

$$f = \{(a, f(a)) \mid a \in A\} \quad \text{or} \quad f = \{(a, b) \in A \times B \mid b = f(a)\}.$$

However, we also learned that some sets of ordered pairs cannot be used to define a function. We now wish to explore under what conditions a set of ordered pairs can be used to define a function. Starting with a function $f\colon A \to B$, since $\text{dom}(f) = A$, we know that

$$\text{For every } a \in A, \text{ there exists a } b \in B \text{ such that } (a, b) \in f. \tag{1}$$

Specifically, we use $b = f(a)$. This says that every element of A can be used as an input. In addition, to be a function, each input can produce only one output. In terms of ordered pairs, this means that there will never be two ordered pairs (a, b) and (a, c) in the function f where $a \in A$, $b, c \in B$, and $b \neq c$. We can formulate this as a conditional statement as follows:

$$\begin{gathered}\text{For every } a \in A \text{ and every } b, c \in B,\\ \text{if } (a, b) \in f \text{ and } (a, c) \in f, \text{ then } b = c.\end{gathered} \tag{2}$$

This also means that if we start with a subset f of $A \times B$ that satisfies conditions (1) and (2), then we can consider f to be a function from A to B by using $b = f(a)$ whenever (a, b) is in f. This proves the following theorem.

Theorem 6.22. *Let A and B be nonempty sets and let f be a subset of $A \times B$ that satisfies the following two properties:*

- *For every $a \in A$, there exists $b \in B$ such that $(a, b) \in f$; and*
- *For every $a \in A$ and every $b, c \in B$, if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$.*

If we use $f(a) = b$ whenever $(a, b) \in f$, then f is a function from A to B.

A Note about Theorem 6.22. The first condition in Theorem 6.22 means that every element of A is an input, and the second condition ensures that every input has exactly one output. Many texts will use Theorem 6.22 as the definition of a function. Many mathematicians believe that this ordered pair representation of a function is the most rigorous definition of a function. It allows us to use set theory to work with and compare functions. For example, equality of functions becomes a question of equality of sets. Therefore, many textbooks will use the ordered pair representation of a function as the definition of a function.

Progress Check 6.23 (Sets of Ordered Pairs that Are Not Functions)
Let $A = \{1, 2, 3\}$ and let $B = \{a, b, c\}$. Explain why each of the following subsets of $A \times B$ cannot be used to define a function from A to B.

1. $F = \{(1, a), (2, a)\}$.
2. $G = \{(1, a), (2, b), (3, c), (2, c)\}$.

The Inverse of a Function

In previous mathematics courses, we learned that the exponential function (with base e) and the natural logarithm function are inverses of each other. This was often expressed as follows:

For each $x \in \mathbb{R}$ with $x > 0$ and for each $y \in \mathbb{R}$,
$y = \ln x$ if and only if $x = e^y$.

Notice that this means that x is the input and y is the output for the natural logarithm function if and only if y is the input and x is the output for the exponential function. In essence, the inverse function (in this case, the exponential function) reverses the action of the original function (in this case, the natural logarithm function). In terms of ordered pairs (input-output pairs), this means that if (x, y) is an ordered pair for a function, then (y, x) is an ordered pair for its inverse. This idea of reversing the roles of the first and second coordinates is the basis for our definition of the inverse of a function.

Definition. Let $f: A \to B$ be a function. The **inverse of f**, denoted by f^{-1}, is the set of ordered pairs $\{(b, a) \in B \times A \mid f(a) = b\}$. That is,

$$f^{-1} = \{(b, a) \in B \times A \mid f(a) = b\}.$$

If we use the ordered pair representation for f, we could also write

$$f^{-1} = \{(b, a) \in B \times A \mid (a, b) \in f\}.$$

Notice that this definition does not state that f^{-1} is a function. It is simply a subset of $B \times A$. After we study the material in Chapter 7, we will say that this means that f^{-1} is a **relation** from B to A. This fact, however, is not important to us now. We are mainly interested in the following question:

Under what conditions will the inverse of the function $f: A \to B$ be a function from B to A?

Progress Check 6.24 (Exploring the Inverse of a Function)
Let $A = \{a, b, c\}$, $B = \{a, b, c, d\}$, and $C = \{p, q, r\}$. Define

$f: A \to C$ by	$g: A \to C$ by	$h: B \to C$ by
$f(a) = r$	$g(a) = p$	$h(a) = p$
$f(b) = p$	$g(b) = q$	$h(b) = q$
$f(c) = q$	$g(c) = p$	$h(c) = r$
		$h(d) = q$

1. Draw an arrow diagram for each function.

2. Determine the inverse of each function as a set of ordered pairs.

3. (a) Is f^{-1} a function from C to A? Explain.
   (b) Is g^{-1} a function from C to A? Explain.
   (c) Is h^{-1} a function from C to B? Explain.

4. Draw an arrow diagram for each inverse from Part (3) that is a function. Use your existing arrow diagram from Part (1) to draw this arrow diagram.

5. Make a conjecture about what conditions on a function $F: S \to T$ will ensure that its inverse is a function from T to S.

We will now consider a general argument suggested by the explorations in Progress Check 6.24. By definition, if $f: A \to B$ is a function, then f^{-1} is a subset of $B \times A$. However, f^{-1} may or may not be a function from B to A. For example, suppose that $s, t \in A$ with $s \neq t$ and $f(s) = f(t)$. This is represented in Figure 6.9.

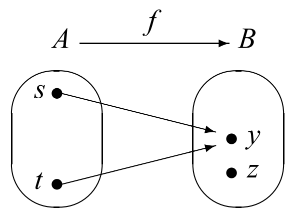


Figure 6.9: The Inverse Is Not a Function

In this case, if we try to reverse the arrows, we will not get a function from B to A. This is because $(y, s) \in f^{-1}$ and $(y, t) \in f^{-1}$ with $s \neq t$. Consequently, f^{-1} is not a function. This suggests that when f is not an injection, then f^{-1} is not a function.

Also, if f is not a surjection, then there exists a $z \in B$ such that $f(a) \neq z$ for all $a \in A$, as in the diagram in Figure 6.9. In other words, there is no ordered pair in f with z as the second coordinate. This means that there would be no ordered pair in f^{-1} with z as a first coordinate. Consequently, f^{-1} cannot be a function from B to A.

This motivates the statement in Theorem 6.25. In the proof of this theorem, we will frequently change back and forth from the input-output representation of a function and the ordered pair representation of a function. The idea is that if $G: S \to T$ is a function, then for $s \in S$ and $t \in T$,

$$G(s) = t \text{ if and only if } (s, t) \in G.$$

When we use the ordered pair representation of a function, we will also use the ordered pair representation of its inverse. In this case, we know that

$$(s, t) \in G \text{ if and only if } (t, s) \in G^{-1}.$$

Theorem 6.25. *Let A and B be nonempty sets and let $f: A \to B$. The inverse of f is a function from B to A if and only if f is a bijection.*

Proof. Let A and B be nonempty sets and let $f: A \to B$. We will first assume that f is a bijection and prove that f^{-1} is a function from B to A. To do this, we will show that f^{-1} satisfies the two conditions of Theorem 6.22.

We first choose $b \in B$. Since the function f is a surjection, there exists an $a \in A$ such that $f(a) = b$. This implies that $(a, b) \in f$ and hence that $(b, a) \in f^{-1}$. Thus, each element of B is the first coordinate of an ordered pair in f^{-1}, and hence f^{-1} satisfies the first condition of Theorem 6.22.

To prove that f^{-1} satisfies the second condition of Theorem 6.22, we must show that each element of B is the first coordinate of exactly one ordered pair in f^{-1}. So let $b \in B$, $a_1, a_2 \in A$ and assume that

$$(b, a_1) \in f^{-1} \text{ and } (b, a_2) \in f^{-1}.$$

This means that $(a_1, b) \in f$ and $(a_2, b) \in f$. We can then conclude that

$$f(a_1) = b \text{ and } f(a_2) = b.$$

But this means that $f(a_1) = f(a_2)$. Since f is a bijection, it is an injection, and we can conclude that $a_1 = a_2$. This proves that b is the first element of only one ordered pair in f^{-1}. Consequently, we have proved that f^{-1} satisfies both conditions of Theorem 6.22 and hence that f^{-1} is a function from B to A.

We now assume that f^{-1} is a function from B to A and prove that f is a bijection. First, to prove that f is an injection, we assume that $a_1, a_2 \in A$ and that $f(a_1) = f(a_2)$. We wish to show that $a_1 = a_2$. If we let $b = f(a_1) = f(a_2)$, we can conclude that

$$(a_1, b) \in f \text{ and } (a_2, b) \in f.$$

But this means that

$$(b, a_1) \in f^{-1} \text{ and } (b, a_2) \in f^{-1}.$$

Since we have assumed that f^{-1} is a function, we can conclude that $a_1 = a_2$. Hence, f is an injection.

Now to prove that f is a surjection, we choose $b \in B$ and will show that there exists an $a \in A$ such that $f(a) = b$. Since f^{-1} is a function, b must be the first coordinate of some ordered pair in f^{-1}. Consequently, there exists an $a \in A$ such that

$$(b, a) \in f^{-1}.$$

Now this implies that $(a, b) \in f$ and hence that $f(a) = b$. This proves that f is a surjection. Since we have also proved that f is an injection, we conclude that f is a bijection. ■

Inverse Function Notation

In the situation where $f\colon A \to B$ is a bijection and f^{-1} is a function from B to A, we can write $f^{-1}\colon B \to A$. In this case, we frequently say that f is an **invertible function**, and we usually do not use the ordered pair representation for either f or f^{-1}. Instead of writing $(a, b) \in f$, we write $f(a) = b$, and instead of writing $(b, a) \in f^{-1}$, we write $f^{-1}(b) = a$. Using the fact that $(a, b) \in f$ if and only if $(b, a) \in f^{-1}$, we can now write $f(a) = b$ if and only if $f^{-1}(b) = a$. We summarize this in Theorem 6.26.

Theorem 6.26. *Let A and B be nonempty sets and let $f\colon A \to B$ be a bijection. Then $f^{-1}\colon B \to A$ is a function, and for every $a \in A$ and $b \in B$,*

$$f(a) = b \text{ if and only if } f^{-1}(b) = a.$$

Example 6.27 (Inverse Function Notation)
For an example of the use of the notation in Theorem 6.26, let $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$. Define

$$f\colon \mathbb{R} \to \mathbb{R} \text{ by } f(x) = x^3; \text{ and } g\colon \mathbb{R} \to \mathbb{R}^+ \text{ by } g(x) = e^x.$$

Notice that $\mathbb{R}^+$ is the codomain of g. We can then say that both f and g are bijections. Consequently, the inverses of these functions are also functions. In fact,

$$f^{-1}\colon \mathbb{R} \to \mathbb{R} \text{ by } f^{-1}(y) = \sqrt[3]{y}; \text{ and } g^{-1}\colon \mathbb{R}^+ \to \mathbb{R} \text{ by } g^{-1}(y) = \ln y.$$

For each function (and its inverse), we can write the result of Theorem 6.26 as follows:

Theorem 6.26	**Translates to:**
For $x, y \in \mathbb{R}$, $f(x) = y$ if and only if $f^{-1}(y) = x$.	For $x, y \in \mathbb{R}$, $x^3 = y$ if and only if $\sqrt[3]{y} = x$.
For $x \in \mathbb{R}$, $y \in \mathbb{R}^+$, $g(x) = y$ if and only if $g^{-1}(y) = x$.	For $x \in \mathbb{R}$, $y \in \mathbb{R}^+$, $e^x = y$ if and only if $\ln y = x$.

Theorems about Inverse Functions

The next two results in this section are two important theorems about inverse functions. The first is actually a corollary of Theorem 6.26.

Corollary 6.28. *Let A and B be nonempty sets and let $f: A \to B$ be a bijection. Then*

1. *For every x in A, $\left(f^{-1} \circ f\right)(x) = x$.*
2. *For every y in B, $\left(f \circ f^{-1}\right)(y) = y$.*

Proof. Let A and B be nonempty sets and assume that $f: A \to B$ is a bijection. So let $x \in A$ and let $f(x) = y$. By Theorem 6.26, we can conclude that $f^{-1}(y) = x$. Therefore,

$$\begin{aligned}\left(f^{-1} \circ f\right)(x) &= f^{-1}(f(x)) \\ &= f^{-1}(y) \\ &= x.\end{aligned}$$

Hence, for each $x \in A$, $\left(f^{-1} \circ f\right)(x) = x$.

The proof that for each y in B, $\left(f \circ f^{-1}\right)(y) = y$ is Exercise (4). ■

Example 6.27 (continued)
For the cubing function and the cube root function, we have seen that

$$\text{For } x, y \in \mathbb{R},\ x^3 = y \text{ if and only if } \sqrt[3]{y} = x.$$

Notice that

- If we substitute $x^3 = y$ into the equation $\sqrt[3]{y} = x$, we obtain $\sqrt[3]{x^3} = x$.
- If we substitute $\sqrt[3]{y} = x$ into the equation $x^3 = y$, we obtain $\left(\sqrt[3]{y}\right)^3 = y$.

This is an illustration of Corollary 6.28. We can see this by using $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ and $f^{-1}: \mathbb{R} \to \mathbb{R}$ defined by $f^{-1}(y) = \sqrt[3]{y}$. Then $f^{-1} \circ f: \mathbb{R} \to \mathbb{R}$ and $f^{-1} \circ f = I_{\mathbb{R}}$. So for each $x \in \mathbb{R}$,

$$\begin{aligned}\left(f^{-1} \circ f\right)(x) &= x \\ f^{-1}(f(x)) &= x \\ f^{-1}\left(x^3\right) &= x \\ \sqrt[3]{x^3} &= x.\end{aligned}$$

Similarly, the equation $\left(\sqrt[3]{y}\right)^3 = y$ for each $y \in \mathbb{R}$ can be obtained from the fact that for each $y \in \mathbb{R}$, $(f \circ f^{-1})(y) = y$.

We will now consider the case where $f\colon A \to B$ and $g\colon B \to C$ are both bijections. In this case, $f^{-1}\colon B \to A$ and $g^{-1}\colon C \to B$. Figure 6.10 can be used to illustrate this situation.

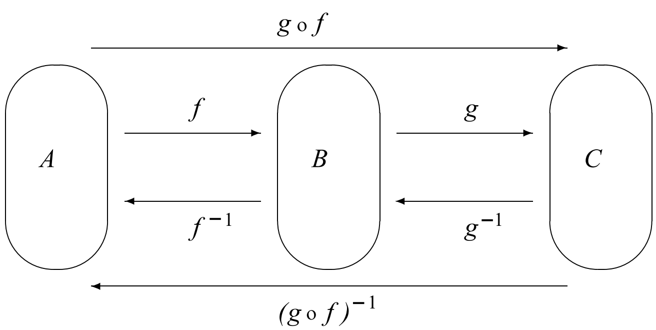


Figure 6.10: Composition of Two Bijections

By Theorem 6.20, $g \circ f\colon A \to C$ is also a bijection. Hence, by Theorem 6.25, $(g \circ f)^{-1}$ is a function and, in fact, $(g \circ f)^{-1}\colon C \to A$. Notice that we can also form the composition of g^{-1} followed by f^{-1} to get $f^{-1} \circ g^{-1}\colon C \to A$. Figure 6.10 helps illustrate the result of the next theorem.

Theorem 6.29. *Let $f\colon A \to B$ and $g\colon B \to C$ be bijections. Then $g \circ f$ is a bijection and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

Proof. Let $f\colon A \to B$ and $g\colon B \to C$ be bijections. Then $f^{-1}\colon B \to A$ and $g^{-1}\colon C \to B$. Hence, $f^{-1} \circ g^{-1}\colon C \to A$. Also, by Theorem 6.20, $g \circ f\colon A \to C$ is a bijection, and hence $(g \circ f)^{-1}\colon C \to A$. We will now prove that for each $z \in C$, $(g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)$.

Let $z \in C$. Since the function g is a surjection, there exists a $y \in B$ such that

$$g(y) = z. \tag{1}$$

Also, since f is a surjection, there exists an $x \in A$ such that

$$f(x) = y. \tag{2}$$

Now these two equations can be written in terms of the respective inverse functions as

$$g^{-1}(z) = y; \text{ and} \tag{3}$$
$$f^{-1}(y) = x. \tag{4}$$

Using equations (3) and (4), we see that

$$\begin{aligned}\left(f^{-1} \circ g^{-1}\right)(z) &= f^{-1}\left(g^{-1}(z)\right) \\ &= f^{-1}(y) \\ &= x. \end{aligned} \tag{5}$$

Using equations (1) and (2) again, we see that $(g \circ f)(x) = z$. However, in terms of the inverse function, this means that

$$(g \circ f)^{-1}(z) = x. \tag{6}$$

Comparing equations (5) and (6), we have shown that for all $z \in C$, $(g \circ f)^{-1}(z) = (f^{-1} \circ g^{-1})(z)$. This proves that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. ■

Exercises 6.5

1. Let $A = \{1, 2, 3\}$ and $B = \{a, b, c\}$.

(**a**) Construct an example of a function $f: A \to B$ that is not a bijection. Write the inverse of this function as a set of ordered pairs. Is the inverse of f a function? Explain. If so, draw an arrow diagram for f and f^{-1}.

(**b**) Construct an example of a function $g: A \to B$ that is a bijection. Write the inverse of this function as a set of ordered pairs. Is the inverse of g a function? Explain. If so, draw an arrow diagram for g and g^{-1}.

2. Let $S = \{a, b, c, d\}$. Define $f: S \to S$ by defining f to be the following set of ordered pairs.

$$f = \{(a, c), (b, b), (c, d), (d, a)\}$$

(**a**) Draw an arrow diagram to represent the function f. Is the function f a bijection?

⋆ **(b)** Write the inverse of f as a set of ordered pairs. Is f^{-1} a function? Explain.

(c) Draw an arrow diagram for f^{-1} using the arrow diagram from Exercise (2a).

⋆ **(d)** Compute $\left(f^{-1} \circ f\right)(x)$ and $\left(f \circ f^{-1}\right)(x)$ for each x in S. What theorem does this illustrate?

⋆ **3.** Inverse functions can be used to help solve certain equations. The idea is to use an inverse function to undo the function.

(a) Since the cube root function and the cubing function are inverses of each other, we can often use the cube root function to help solve an equation involving a cube. For example, the main step in solving the equation

$$(2t - 1)^3 = 20$$

is to take the cube root of each side of the equation. This gives

$$\begin{aligned} \sqrt[3]{(2t-1)^3} &= \sqrt[3]{20} \\ 2t - 1 &= \sqrt[3]{20}. \end{aligned}$$

Explain how this step in solving the equation is a use of Corollary 6.28.

(b) A main step in solving the equation $e^{2t-1} = 20$ is to take the natural logarithm of both sides of this equation. Explain how this step is a use of Corollary 6.28, and then solve the resulting equation to obtain a solution for t in terms of the natural logarithm function.

(c) How are the methods of solving the equations in Exercise (3a) and Exercise (3b) similar?

⋆ **4.** Prove Part (2) of Corollary 6.28. Let A and B be nonempty sets and let $f: A \to B$ be a bijection. Then for every y in B, $\left(f \circ f^{-1}\right)(y) = y$.

5. In Progress Check 6.6 on page 298, we defined the identity function on a set. The **identity function on the set T**, denoted by I_T, is the function $I_T: T \to T$ defined by $I_T(t) = t$ for every t in T. Explain how Corollary 6.28 can be stated using the concept of equality of functions and the identity functions on the sets A and B.

6. Let $f: A \to B$ and $g: B \to A$. Let I_A and I_B be the identity functions on the sets A and B, respectively. Prove each of the following:

* ⋆ **(a)** If $g \circ f = I_A$, then f is an injection.
* ⋆ **(b)** If $f \circ g = I_B$, then f is a surjection.
* **(c)** If $g \circ f = I_A$ and $f \circ g = I_B$, then f and g are bijections and $g = f^{-1}$.

⋆ **7.** **(a)** Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = e^{-x^2}$. Is the inverse of f a function? Justify your conclusion.

(b) Let $\mathbb{R}^* = \{x \in \mathbb{R} \mid x \geq 0\}$. Define $g : \mathbb{R}^* \to (0, 1]$ by $g(x) = e^{-x^2}$. Is the inverse of g a function? Justify your conclusion.

8. **(a)** Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Explain why the inverse of f is not a function.

(b) Let $\mathbb{R}^* = \{t \in \mathbb{R} \mid t \geq 0\}$. Define $g : \mathbb{R}^* \to \mathbb{R}^*$ by $g(x) = x^2$. Explain why this squaring function (with a restricted domain and codomain) is a bijection.

(c) Explain how to define the square root function as the inverse of the function in Exercise (8b).

(d) True or false: $\left(\sqrt{x}\right)^2 = x$ for all $x \in \mathbb{R}$ such that $x \geq 0$.

(e) True or false: $\sqrt{x^2} = x$ for all $x \in \mathbb{R}$.

9. Prove the following:

If $f : A \to B$ is a bijection, then $f^{-1} : B \to A$ is also a bijection.

10. For each natural number k, let A_k be a set, and for each natural number n, let $f_n : A_n \to A_{n+1}$.

For example, $f_1 : A_1 \to A_2$, $f_2 : A_2 \to A_3$, $f_3 : A_3 \to A_4$, and so on.

Use mathematical induction to prove that for each natural number n with $n \geq 2$, if $f_1, f_2, \ldots, f_n$ are all bijections, then $f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$ is a bijection and

$$(f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1)^{-1} = f_1^{-1} \circ f_2^{-1} \circ \cdots \circ f_{n-1}^{-1} \circ f_n^{-1}.$$

Note: This is an extension of Theorem 6.29. In fact, Theorem 6.29 is the basis step of this proof for $n = 2$.

11. **(a)** Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 - 4$ for all $x \in \mathbb{R}$. Explain why the inverse of the function f is not a function.

(**b**) Let $\mathbb{R}^* = \{x \in \mathbb{R} \mid x \geq 0\}$ and let $T = \{y \in \mathbb{R} \mid y \geq -4\}$. Define $F: \mathbb{R}^* \to T$ by $F(x) = x^2 - 4$ for all $x \in \mathbb{R}^*$. Explain why the inverse of the function F is a function and find a formula for $F^{-1}(y)$, where $y \in T$.

12. Let $R_5 = \{0, 1, 2, 3, 4\}$.

(**a**) Define $f: R_5 \to R_5$ by $f(x) = x^2 + 4 \pmod 5$ for all $x \in R_5$. Write the inverse of f as a set of ordered pairs and explain why f^{-1} is not a function.

(**b**) Define $g: R_5 \to R_5$ by $g(x) = x^3 + 4 \pmod 5$ for all $x \in R_5$. Write the inverse of g as a set of ordered pairs and explain why g^{-1} is a function.

(**c**) Is it possible to write a formula for $g^{-1}(y)$, where $y \in R_5$? The answer to this question depends on whether or not it is possible to define a cube root of elements of R_5. Recall that for a real number x, we define the cube root of x to be the real number y such that $y^3 = x$. That is,

$$y = \sqrt[3]{x} \text{ if and only if } y^3 = x.$$

Using this idea, is it possible to define the cube root of each number in R_5? If so, what are $\sqrt[3]{0}$, $\sqrt[3]{1}$, $\sqrt[3]{2}$, $\sqrt[3]{3}$, and $\sqrt[3]{4}$?

(**d**) Now answer the question posed at the beginning of Part (c). If possible, determine a formula for $g^{-1}(y)$ where $g^{-1}: R_5 \to R_5$.

Explorations and Activities

13. Constructing an Inverse Function. If $f: A \to B$ is a bijection, then we know that its inverse is a function. If we are given a formula for the function f, it may be desirable to determine a formula for the function f^{-1}. This can sometimes be done, while at other times it is very difficult or even impossible.

Let $f: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 2x^3 - 7$. A graph of this function would suggest that this function is a bijection.

(**a**) Prove that the function f is an injection and a surjection.

Let $y \in \mathbb{R}$. One way to prove that f is a surjection is to set $y = f(x)$ and solve for x. If this can be done, then we would know that there exists an $x \in \mathbb{R}$ such that $f(x) = y$. For the function f, we are using x for the input and y for the output. By solving for x in terms of y, we are attempting to write a formula where y is the input and x is the output. This formula represents the inverse function.

(**b**) Solve the equation $y = 2x^3 - 7$ for x. Use this to write a formula for $f^{-1}(y)$, where $f^{-1}\colon \mathbb{R} \to \mathbb{R}$.

(**c**) Use the result of Part (13b) to verify that for each $x \in \mathbb{R}$, $f^{-1}(f(x)) = x$ and for each $y \in \mathbb{R}$, $f\left(f^{-1}(y)\right) = y$.

Now let $\mathbb{R}^+ = \{y \in \mathbb{R} \mid y > 0\}$. Define $g\colon \mathbb{R} \to \mathbb{R}^+$ by $g(x) = e^{2x-1}$.

(**d**) Set $y = e^{2x-1}$ and solve for x in terms of y.

(**e**) Use your work in Exercise (13d) to define a function $h\colon \mathbb{R}^+ \to \mathbb{R}$.

(**f**) For each $x \in \mathbb{R}$, determine $(h \circ g)(x)$ and for each $y \in \mathbb{R}^+$, determine $(g \circ h)(y)$.

(**g**) Use Exercise (6) to explain why $h = g^{-1}$.

14. The Inverse Sine Function. We have seen that in order to obtain an inverse function, it is sometimes necessary to restrict the domain (or the codomain) of a function.

(**a**) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \sin x$. Explain why the inverse of the function f is not a function. (A graph may be helpful.)

Notice that if we use the ordered pair representation, then the sine function can be represented as

$$f = \{(x, y) \in \mathbb{R} \times \mathbb{R} \mid y = \sin x\}.$$

If we denote the inverse of the sine function by $\sin^{-1}$, then

$$f^{-1} = \{(y, x) \in \mathbb{R} \times \mathbb{R} \mid y = \sin x\}.$$

Part (14a) proves that f^{-1} is not a function. However, in previous mathematics courses, we frequently used the "inverse sine function." This is not really the inverse of the sine function as defined in Part (14a) but, rather, it is the inverse of the sine function **restricted to the domain** $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

(**b**) Explain why the function $F\colon \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] \to [-1, 1]$ defined by $F(x) = \sin x$ is a bijection.

The inverse of the function in Part (14b) is itself a function and is called the **inverse sine function** (or sometimes the **arcsine function**).

(**c**) What is the domain of the inverse sine function? What are the range and codomain of the inverse sine function?

Let us now use $F(x) = \text{Sin}(x)$ to represent the restricted sine function in Part (14b). Therefore, $F^{-1}(x) = \text{Sin}^{-1}(x)$ can be used to represent the inverse sine function. Observe that

$$F\colon \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1] \text{ and } F^{-1}\colon [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

(**d**) Using this notation, explain why

$\text{Sin}^{-1} y = x$ if and only if $\left[y = \sin x \text{ and } -\dfrac{\pi}{2} \le x \le \dfrac{\pi}{2}\right]$;

$\text{Sin}\left(\text{Sin}^{-1}(y)\right) = y$ for all $y \in [-1, 1]$; and

$\text{Sin}^{-1}(\text{Sin}(x)) = x$ for all $x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$.

6.6 Functions Acting on Sets

Preview Activity 1 (Functions and Sets)

Let $S = \{a, b, c, d\}$ and $T = \{s, t, u\}$. Define $f\colon S \to T$ by

$$f(a) = s \qquad f(b) = t \qquad f(c) = t \qquad f(d) = s.$$

1. Let $A = \{a, c\}$ and $B = \{a, d\}$. Notice that A and B are subsets of S. Use the roster method to specify the elements of the following two subsets of T:

   (**a**) $\{f(x) \mid x \in A\}$ (**b**) $\{f(x) \mid x \in B\}$

2. Let $C = \{s, t\}$ and $D = \{s, u\}$. Notice that C and D are subsets of T. Use the roster method to specify the elements of the following two subsets of S:

   (**a**) $\{x \in S \mid f(x) \in C\}$ (**b**) $\{x \in S \mid f(x) \in D\}$

Now let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2$, for each $x \in \mathbb{R}$.

3. Let $A = \{1, 2, 3, -1\}$. Use the roster method to specify the elements of the set $\{g(x) \mid x \in A\}$.

4. Use the roster method to specify the elements of each of the following sets:

(**a**) $\{x \in \mathbb{R} \mid g(x) = 1\}$

(**b**) $\{x \in \mathbb{R} \mid g(x) = 9\}$

(**c**) $\{x \in \mathbb{R} \mid g(x) = 15\}$

(**d**) $\{x \in \mathbb{R} \mid g(x) = -1\}$

5. Let $B = \{1, 9, 15, -1\}$. Use the roster method to specify the elements of the set $\{x \in \mathbb{R} \mid g(x) \in B\}$.

Preview Activity 2 (**Functions and Intervals**)
Let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = x^2$, for each $x \in \mathbb{R}$.

1. We will first determine where g maps the closed interval $[1, 2]$. (Recall that $[1, 2] = \{x \in \mathbb{R} \mid 1 \leq x \leq 2\}$.) That is, we will describe, in simpler terms, the set $\{g(x) \mid x \in [1, 2]\}$. This is the set of all images of the real numbers in the closed interval $[1, 2]$.

(**a**) Draw a graph of the function g using $-3 \leq x \leq 3$.

(**b**) On the graph, draw the vertical lines $x = 1$ and $x = 2$ from the x-axis to the graph. Label the points $P(1, f(1))$ and $Q(2, f(2))$ on the graph.

(**c**) Now draw horizontal lines from the points P and Q to the y-axis. Use this information from the graph to describe the set $\{g(x) \mid x \in [1, 2]\}$ in simpler terms. Use interval notation or set builder notation.

2. We will now determine all real numbers that g maps into the closed interval $[1, 4]$. That is, we will describe the set $\{x \in \mathbb{R} \mid g(x) \in [1, 4]\}$ in simpler terms. This is the set of all preimages of the real numbers in the closed interval $[1, 4]$.

(**a**) Draw a graph of the function g using $-3 \leq x \leq 3$.

(**b**) On the graph, draw the horizontal lines $y = 1$ and $y = 4$ from the y-axis to the graph. Label all points where these two lines intersect the graph.

(**c**) Now draw vertical lines from the points in Part (2) to the x-axis, and then use the resulting information to describe the set $\{x \in \mathbb{R} \mid g(x) \in [1, 4]\}$ in simpler terms. (You will need to describe this set as a union of two intervals. Use interval notation or set builder notation.)

Functions Acting on Sets

In our study of functions, we have focused on how a function "maps" individual elements of its domain to the codomain. We also studied the preimage of an individual element in its codomain. For example, if $f: \mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^2$, for each $x \in \mathbb{R}$, then

- $f(2) = 4$. We say that f maps 2 to 4 or that 4 is the image of 2 under the function f.
- Since $f(x) = 4$ implies that $x = 2$ or $x = -2$, we say that the preimages of 4 are 2 and -2 or that the set of preimages of 4 is $\{-2, 2\}$.

For a function $f: S \to T$, the next step is to consider subsets of S or T and what corresponds to them in the other set. We did this in the preview activities. We will give some definitions and then revisit the examples in the preview activities in light of these definitions. We will first consider the situation where A is a subset of S and consider the set of outputs whose inputs are from A. This will be a subset of T.

Definition. Let $f: S \to T$. If $A \subseteq S$, then the **image of A under f** is the set $f(A)$, where

$$f(A) = \{f(x) \mid x \in A\}.$$

If there is no confusion as to which function is being used, we call $f(A)$ **the image of A**.

We now consider the situation in which C is a subset of T and consider the subset of A consisting of all elements of T whose outputs are in C.

Definition. Let $f: S \to T$. If $C \subseteq T$, then the **preimage of C under f** is the set $f^{-1}(C)$, where

$$f^{-1}(C) = \{x \in S \mid f(x) \in C\}.$$

If there is no confusion as to which function is being used, we call $f^{-1}(C)$ **the preimage of C**. The preimage of the set C under f is also called the **inverse image of C under f**.

Notice that the set $f^{-1}(C)$ is defined whether or not f^{-1} is a function.

Progress Check 6.30 (Preview Activity 1 Revisited)
Let $S = \{a, b, c, d\}$ and $T = \{s, t, u\}$. Define $f : S \to T$ by

$$f(a) = s \qquad f(b) = t \qquad f(c) = t \qquad f(d) = s.$$

Let $\quad A = \{a, c\}, \quad B = \{a, d\}, \quad C = \{s, t\}, \quad$ and $\quad D = \{s, u\}$.

Use your work in Preview Activity 1 to determine each of the following sets:

1. $f(A)$ **2.** $f(B)$ **3.** $f^{-1}(C)$ **4.** $f^{-1}(D)$

Example 6.31 (Images and Preimages of Sets)
Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$, for each $x \in \mathbb{R}$. The following results are based on the examples in Preview Activity 1 and Preview Activity 2.

- Let $A = \{1, 2, 3, -1\}$. Then $f(A) = \{1, 4, 9\}$.
- Let $B = \{1, 9, 15, -1\}$. Then $f^{-1}(B) = \left\{-\sqrt{15}, -3, -1, 1, 3, \sqrt{15}\right\}$.

The graphs from Preview Activity 2 illustrate the following results:

- If T is the closed interval $[1, 2]$, then the image of the set T is

$$\begin{aligned} f(T) &= \{f(x) \mid x \in [1, 2]\} \\ &= [1, 4]. \end{aligned}$$

- If C is the closed interval $[1, 4]$, then the preimage of the set C is

$$f^{-1}(C) = \{x \in \mathbb{R} \mid f(x) \in [1, 4]\} \quad = [-2, -1] \cup [1, 2].$$

Set Operations and Functions Acting on Sets

We will now consider the following situation: Let S and T be sets and let f be a function from S to T. Also, let A and B be subsets of S and let C and D be subsets of T. In the remainder of this section, we will consider the following situations and answer the questions posed in each case.

- The set $A \cap B$ is a subset of S and so $f(A \cap B)$ is a subset of T. In addition, $f(A)$ and $f(B)$ are subsets of T. Hence, $f(A) \cap f(B)$ is a subset of T.

  Is there any relationship between $f(A \cap B)$ and $f(A) \cap f(B)$?

- The set $A \cup B$ is a subset of S and so $f(A \cup B)$ is a subset of T. In addition, $f(A)$ and $f(B)$ are subsets of T. Hence, $f(A) \cup f(B)$ is a subset of T.

  Is there any relationship between $f(A \cup B)$ and $f(A) \cup f(B)$?

- The set $C \cap D$ is a subset of T and so $f^{-1}(C \cap D)$ is a subset of S. In addition, $f^{-1}(C)$ and $f^{-1}(D)$ are subsets of S. Hence, $f^{-1}(C) \cap f^{-1}(D)$ is a subset of S.

  Is there any relationship between the sets $f^{-1}(C \cap D)$ and $f^{-1}(C) \cap f^{-1}(D)$?

- The set $C \cup D$ is a subset of T and so $f^{-1}(C \cup D)$ is a subset of S. In addition, $f^{-1}(C)$ and $f^{-1}(D)$ are subsets of S. Hence, $f^{-1}(C) \cup f^{-1}(D)$ is a subset of S.

  Is there any relationship between the sets $f^{-1}(C \cup D)$ and $f^{-1}(C) \cup f^{-1}(D)$?

These and other questions will be explored in the next progress check.

Progress Check 6.32 (**Set Operations and Functions Acting on Sets**)
In Section 6.2, we introduced functions involving congruences. For example, if we let

$$R_8 = \{0, 1, 2, 3, 4, 5, 6, 7\},$$

then we can define $f\colon R_8 \to R_8$ by $f(x) = r$, where $\left(x^2 + 2\right) \equiv r \pmod 8$ and $r \in R_8$. Moreover, we shortened this notation to

$$f(x) = \left(x^2 + 2\right) \pmod 8.$$

We will use the following subsets of $\mathbb{R}_8$:

$$A = \{1, 2, 4\} \qquad B = \{3, 4, 6\} \qquad C = \{1, 2, 3\} \qquad D = \{3, 4, 5\}.$$

1. Verify that $f(0) = 2$, $f(1) = 3$, $f(2) = 6$, and $f(3) = 3$. Then determine $f(4)$, $f(5)$, $f(6)$, and $f(7)$.

2. Determine $f(A)$, $f(B)$, $f^{-1}(C)$, and $f^{-1}(D)$.

3. For each of the following, determine the two subsets of R_8 and then determine if there is a relationship between the two sets. For example, $A \cap B = \{4\}$ and since $f(4) = 2$, we see that $f(A \cap B) = \{2\}$.

   (a) $f(A \cap B)$ and $f(A) \cap f(B)$
   (b) $f(A \cup B)$ and $f(A) \cup f(B)$
   (c) $f^{-1}(C \cap D)$ and $f^{-1}(C) \cap f^{-1}(D)$
   (d) $f^{-1}(C \cup D)$ and $f^{-1}(C) \cup f^{-1}(D)$

4. Notice that $f(A)$ is a subset of the codomain, R_8. Consequently, $f^{-1}(f(A))$ is a subset of the domain, R_8. Is there any relation between A and $f^{-1}(f(A))$ in this case?

5. Notice that $f^{-1}(C)$ is a subset of the domain, R_8. Consequently, $f\left(f^{-1}(C)\right)$ is a subset of the codomain, R_8. Is there any relation between C and $f\left(f^{-1}(C)\right)$ in this case?

Example 6.33 (Set Operations and Functions Acting on Sets)
Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^2 + 2$ for all $x \in \mathbb{R}$. It will be helpful to use the graph shown in Figure 6.11.

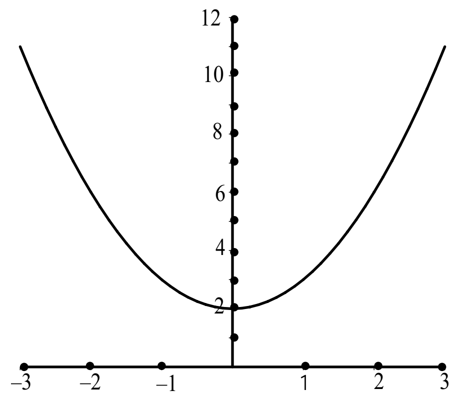


Figure 6.11: Graph for Example 6.33

We will use the following closed intervals:

$$A = [0, 3] \qquad B = [-2, 1] \qquad C = [2, 6] \qquad D = [0, 3].$$

1. Verify that $f(A) = [2, 11]$, $f(B) = [2, 6]$, $f^{-1}(C) = [-2, 2]$, and that $f^{-1}(D) = [-1, 1]$.

2. (**a**) Explain why $f(A \cap B) = [2, 3]$ and $f(A) \cap f(B) = [2, 6]$. So in this case, $f(A \cap B) \subseteq f(A) \cap f(B)$.

   (**b**) Explain why $f(A \cup B) = [2, 11]$ and $f(A) \cup f(B) = [2, 11]$. So in this case, $f(A \cup B) = f(A) \cup f(B)$.

   (**c**) Explain why $f^{-1}(C \cap D) = [-1, 1]$ and $f^{-1}(C) \cap f^{-1}(D) = [-1, 1]$. So in this case, $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

   (**d**) Explain why $f^{-1}(C \cup D) = [-2, 2]$ and $f^{-1}(C) \cup f^{-1}(D) = [-2, 2]$. So in this case, $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$.

3. Recall that $A = [0, 3]$. Notice $f(A) = [2, 11]$ is a subset of the codomain, $\mathbb{R}$. Explain why $f^{-1}(f(A)) = [-3, 3]$. Since $f^{-1}(f(A))$ is a subset of the domain, $\mathbb{R}$, we see that in this case, $A \subseteq f^{-1}(f(A))$.

4. Recall that $C = [2, 6]$. Notice that $f^{-1}(C) = [-2, 2]$ is a subset of the domain, $\mathbb{R}$. Explain why $f\left(f^{-1}(C)\right) = [2, 6]$. Since $f\left(f^{-1}(C)\right)$ is a subset of the codomain, $\mathbb{R}$, we see that in this case $f\left(f^{-1}(C)\right) = C$.

The examples in Progress Check 6.32 and Example 6.33 were meant to illustrate general results about how functions act on sets. In particular, we investigated how the action of a function on sets interacts with the set operations of intersection and union. We will now state the theorems that these examples were meant to illustrate. Some of the proofs will be left as exercises.

Theorem 6.34. *Let $f: S \to T$ be a function and let A and B be subsets of S. Then*

1. $f(A \cap B) \subseteq f(A) \cap f(B)$
2. $f(A \cup B) = f(A) \cup f(B)$

Proof. We will prove Part (1). The proof of Part (2) is Exercise (5).

Assume that $f: S \to T$ is a function and let A and B be subsets of S. We will prove that $f(A \cap B) \subseteq f(A) \cap f(B)$ by proving that for all $y \in T$, if $y \in f(A \cap B)$, then $y \in f(A) \cap f(B)$.

We assume that $y \in f(A \cap B)$. This means that there exists an $x \in A \cap B$ such that $f(x) = y$. Since $x \in A \cap B$, we conclude that $x \in A$ and $x \in B$.

- Since $x \in A$ and $f(x) = y$, we conclude that $y \in f(A)$.
- Since $x \in B$ and $f(x) = y$, we conclude that $y \in f(B)$.

Since $y \in f(A)$ and $y \in f(B)$, $y \in f(A) \cap f(B)$. This proves that if $y \in f(A \cap B)$, then $y \in f(A) \cap f(B)$. Hence $f(A \cap B) \subseteq f(A) \cap f(B)$. ■

Theorem 6.35. *Let $f: S \to T$ be a function and let C and D be subsets of T. Then*

1. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$
2. $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$

Proof. We will prove Part (2). The proof of Part (1) is Exercise (6).

Assume that $f: S \to T$ is a function and that C and D are subsets of T. We will prove that $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$ by proving that each set is a subset of the other.

We start by letting x be an element of $f^{-1}(C \cup D)$. This means that $f(x)$ is an element of $C \cup D$. Hence,

$$f(x) \in C \text{ or } f(x) \in D.$$

In the case where $f(x) \in C$, we conclude that $x \in f^{-1}(C)$, and hence that $x \in f^{-1}(C) \cup f^{-1}(D)$. In the case where $f(x) \in D$, we see that $x \in f^{-1}(D)$, and hence that $x \in f^{-1}(C) \cup f^{-1}(D)$. So in both cases, $x \in f^{-1}(C) \cup f^{-1}(D)$, and we have proved that $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$.

We now let $t \in f^{-1}(C) \cup f^{-1}(D)$. This means that

$$t \in f^{-1}(C) \text{ or } t \in f^{-1}(D).$$

- In the case where $t \in f^{-1}(C)$, we conclude that $f(t) \in C$ and hence that $f(t) \in C \cup D$. This means that $t \in f^{-1}(C \cup D)$.
- Similarly, when $t \in f^{-1}(D)$, it follows that $f(t) \in D$ and hence that $f(t) \in C \cup D$. This means that $t \in f^{-1}(C \cup D)$.

These two cases prove that if $t \in f^{-1}(C) \cup f^{-1}(D)$, then $t \in f^{-1}(C \cup D)$. Therefore, $f^{-1}(C) \cup f^{-1}(D) \subseteq f^{-1}(C \cup D)$.

Since we have now proved that each of the two sets is a subset of the other set, we can conclude that $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$. ■

Theorem 6.36. *Let $f : S \to T$ be a function and let A be a subset of S and let C be a subset of T. Then*

1. $A \subseteq f^{-1}(f(A))$
2. $f\left(f^{-1}(C)\right) \subseteq C$

Proof. We will prove Part (1). The proof of Part (2) is Exercise (7).

To prove Part (1), we will prove that for all $a \in S$, if $a \in A$, then $a \in f^{-1}(f(A))$. So let $a \in A$. Then, by definition, $f(a) \in f(A)$. We know that $f(A) \subseteq T$, and so $f^{-1}(f(A)) \subseteq S$. Notice that

$$f^{-1}(f(A)) = \{x \in S \mid f(x) \in f(A)\}.$$

Since $f(a) \in f(A)$, we use this to conclude that $a \in f^{-1}(f(A))$. This proves that if $a \in A$, then $a \in f^{-1}(f(A))$, and hence that $A \subseteq f^{-1}(f(A))$. ■

Exercises 6.6

1. Let $f : S \to T$, let A and B be subsets of S, and let C and D be subsets of T. For $x \in S$ and $y \in T$, carefully explain what it means to say that

   ⋆ **(a)** $y \in f(A \cap B)$
   (b) $y \in f(A \cup B)$
   (c) $y \in f(A) \cap f(B)$
   ⋆ **(d)** $y \in f(A) \cup f(B)$
   (e) $x \in f^{-1}(C \cap D)$
   ⋆ **(f)** $x \in f^{-1}(C \cup D)$
   (g) $x \in f^{-1}(C) \cap f^{-1}(D)$
   ⋆ **(h)** $x \in f^{-1}(C) \cup f^{-1}(D)$

2. Let $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = -2x + 1$. Let

   $A = [2, 5]$ $\quad B = [-1, 3]$ $\quad C = [-2, 3]$ $\quad D = [1, 4]$.

   Find each of the following:

   (a) $f(A)$
   ⋆ **(b)** $f^{-1}(f(A))$
   (c) $f^{-1}(C)$
   ⋆ **(d)** $f(f^{-1}(C))$
   ⋆ **(e)** $f(A \cap B)$
   ⋆ **(f)** $f(A) \cap f(B)$
   (g) $f^{-1}(C \cap D)$
   (h) $f^{-1}(C) \cap f^{-1}(D)$

3. Let $g\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ by $g(m,n) = 2^m 3^n$, let $A = \{1,2,3\}$, and let $C = \{1,4,6,9,12,16,18\}$. Find

⋆ **(a)** $g(A \times A)$

⋆ **(b)** $g^{-1}(C)$

(c) $g^{-1}(g(A \times A))$

(d) $g\left(g^{-1}(C)\right)$

4. ⋆ **(a)** Let $S = \{1,2,3,4\}$. Define $F\colon S \to \mathbb{N}$ by $F(x) = x^2$ for each $x \in S$. What is the range of the function F and what is $F(S)$? How do these two sets compare?

Now let A and B be sets and let $f\colon A \to B$ be an arbitrary function from A to B.

(b) Explain why $f(A) = \text{range}(f)$.

(c) Define a function $g\colon A \to f(A)$ by $g(x) = f(x)$ for all x in A. Prove that the function g is a surjection.

⋆ **5.** Prove Part (2) of Theorem 6.34.

Let $f\colon S \to T$ be a function and let A and B be subsets of S. Then $f(A \cup B) = f(A) \cup f(B)$.

⋆ **6.** Prove Part (1) of Theorem 6.35.

Let $f\colon S \to T$ be a function and let C and D be subsets of T. Then $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$.

7. Prove Part (2) of Theorem 6.36.

Let $f\colon S \to T$ be a function and let $C \subseteq T$. Then $f\left(f^{-1}(C)\right) \subseteq C$.

8. Let $f\colon S \to T$ and let A and B be subsets of S. Prove or disprove each of the following:

(a) If $A \subseteq B$, then $f(A) \subseteq f(B)$.

(b) If $f(A) \subseteq f(B)$, then $A \subseteq B$.

⋆ **9.** Let $f\colon S \to T$ and let C and D be subsets of T. Prove or disprove each of the following:

(a) If $C \subseteq D$, then $f^{-1}(C) \subseteq f^{-1}(D)$.

(b) If $f^{-1}(C) \subseteq f^{-1}(D)$, then $C \subseteq D$.

10. Prove or disprove:

If $f: S \to T$ is a function and A and B are subsets of S, then $f(A) \cap f(B) \subseteq f(A \cap B)$.

Note: Part (1) of Theorem 6.34 states that $f(A \cap B) \subseteq f(A) \cap f(B)$.

11. Let $f: S \to T$ be a function, let $A \subseteq S$, and let $C \subseteq T$.

(**a**) Part (1) of Theorem 6.36 states that $A \subseteq f^{-1}(f(A))$. Give an example where $f^{-1}(f(A)) \not\subseteq A$.

(**b**) Part (2) of Theorem 6.36 states that $f\left(f^{-1}(C)\right) \subseteq C$. Give an example where $C \not\subseteq f\left(f^{-1}(C)\right)$.

12. Is the following proposition true or false? Justify your conclusion with a proof or a counterexample.

If $f: S \to T$ is an injection and $A \subseteq S$, then $f^{-1}(f(A)) = A$.

13. Is the following proposition true or false? Justify your conclusion with a proof or a counterexample.

If $f: S \to T$ is a surjection and $C \subseteq T$, then $f\left(f^{-1}(C)\right) = C$.

14. Let $f: S \to T$. Prove that $f(A \cap B) = f(A) \cap f(B)$ for all subsets A and B of S if and only if f is an injection.

6.7 Chapter 6 Summary

Important Definitions

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- Image of a set under a function, page 351
- preimage of a set under a function, page 351

Important Theorems and Results about Functions

- **Theorem 6.20**. *Let A, B, and C be nonempty sets and let $f: A \to B$ and $g: B \to C$.*

  1. *If f and g are both injections, then $g \circ f$ is an injection.*
  2. *If f and g are both surjections, then $g \circ f$ is a surjection.*
  3. *If f and g are both bijections, then $g \circ f$ is a bijection.*

- **Theorem 6.21**. *Let A, B, and C be nonempty sets and let $f: A \to B$ and $g: B \to C$.*

  1. *If $g \circ f: A \to C$ is an injection, then $f: A \to B$ is an injection.*
  2. *If $g \circ f: A \to C$ is a surjection, then $g: B \to C$ is a surjection.*

- **Theorem 6.22**. *Let A and B be nonempty sets and let f be a subset of $A \times B$ that satisfies the following two properties:*

  - *For every $a \in A$, there exists $b \in B$ such that $(a, b) \in f$; and*
  - *For every $a \in A$ and every $b, c \in B$, if $(a, b) \in f$ and $(a, c) \in f$, then $b = c$.*

  If we use $f(a) = b$ whenever $(a, b) \in f$, then f is a function from A to B.

- **Theorem 6.25**. *Let A and B be nonempty sets and let $f: A \to B$. The inverse of f is a function from B to A if and only if f is a bijection.*

- **Theorem 6.26**. *Let A and B be nonempty sets and let $f: A \to B$ be a bijection. Then $f^{-1}: B \to A$ is a function, and for every $a \in A$ and $b \in B$,*

$$f(a) = b \text{ if and only if } f^{-1}(b) = a.$$

- **Corollary 6.28**. *Let A and B be nonempty sets and let $f: A \to B$ be a bijection. Then*
  1. *For every x in A, $(f^{-1} \circ f)(x) = x$.*
  2. *For every y in B, $(f \circ f^{-1})(y) = y$.*
- **Theorem 6.29**. *Let $f: A \to B$ and $g: B \to C$ be bijections. Then $g \circ f$ is a bijection and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*
- **Theorem 6.34**. *Let $f: S \to T$ be a function and let A and B be subsets of S. Then*
  1. $f(A \cap B) \subseteq f(A) \cap f(B)$
  2. $f(A \cup B) = f(A) \cup f(B)$
- **Theorem 6.35**. *Let $f: S \to T$ be a function and let C and D be subsets of T. Then*
  1. $f^{-1}(C \cap D) = f^{-1}(C) \cap f^{-1}(D)$
  2. $f^{-1}(C \cup D) = f^{-1}(C) \cup f^{-1}(D)$
- **Theorem 6.36**. *Let $f: S \to T$ be a function and let A be a subset of S and let C be a subset of T. Then*
  1. $A \subseteq f^{-1}(f(A))$
  2. $f\left(f^{-1}(C)\right) \subseteq C$