Section 7

Open Balls and Neighborhoods in Metric Spaces

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is an open ball in a metric space? Give one important property of open balls.
- What is a neighborhood of a point in a metric space?
- How can we use open balls or neighborhoods to determine the continuity of a function at a point?

Introduction

Open sets are vitally important in topology. We will see later that every topological space is completely defined by its open sets, and continuous functions can be defined just in terms of open sets. In this section we introduce the idea of open balls and neighborhoods in metric spaces and discover a few of their properties. This discussion will form the basis for introducing open sets in the next section.

Recall that the continuity of a function f from a metric space (X, d_X) to a metric space (Y, d_Y) at a point a is defined in terms of sets of points $x \in X$ such that $d_X(x, a) < \delta$ and $y \in Y$ such that $d_Y(y, f(a)) < \epsilon$ for positive real numbers δ and ϵ. In $\mathbb{R}$ with the Euclidean metric d_E, for real numbers x and a the set of x values satisfying $d_E(x, a) < \delta$ is the set of x values so that $|x - a| < \delta$. We often write this set in interval notation as $(a - \delta, a + \delta)$ and call $(a - \delta, a + \delta)$ an open interval. An informal reason that we call such an interval open (as opposed to the intervals $[a - \delta, a + \delta)$, $(a - \delta, a + \delta]$, or $[a - \delta, a + \delta]$) is that the open interval does not contain either of its endpoints. A more substantial reason to call such an interval open is that if x' is any element in $(a - \delta, a + \delta)$, then we can find another open interval around x' that is completely contained in the interval $(a - \delta, a + \delta)$. So you could naively think of an open interval as one in which there is enough room in the interval for any point in the interval to wiggle around a bit and stay within the interval.

Since the open interval $(a - \delta, a + \delta)$ can be described completely by the Euclidean metric as the set of x values so that $d_E(x, a) < \delta$, there is no reason why we can't extend this notation of open interval to any metric space. We must note, though, that $\mathbb{R}$ is one-dimensional while most metric spaces are not, so the term "interval" will no longer be appropriate. We replace the concept of interval with that of an open ball.

Definition 7.1. Let (X, d_X) be a metric space, and let $a \in X$. For $\delta > 0$, the **open ball** $B(a, \delta)$ **of radius** δ **around** a is the set

$$B(a, \delta) = \{x \in X \mid d_X(x, a) < \delta\}.$$

We note here that our notation for an open ball is not universal. For example, some texts use $B_\delta(a)$ for our $B(a, \delta)$.

Preview Activity 7.1. Describe and draw a picture of the indicated open ball in each of the following metric spaces.

(1) The open ball $B(2, 1)$ in the metric space $(\mathbb{R}, d_E)$ with the Euclidean metric

$$d_E(x, y) = |x - y|.$$

(2) The open ball $B((3, 2), 1)$ in the metric space $(\mathbb{R}^2, d_E)$ with the Euclidean metric

$$d_E((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

(3) The open ball $B((3, 2), 1)$ in the metric space $(\mathbb{R}^2, d_M)$ with the max metric

$$d_M((x_1, x_2), (y_1, y_2)) = \max\{|x_1 - y_1|, |x_2 - y_2|\}.$$

(4) The open ball $B((3, 2), 1)$ in the metric space $(\mathbb{R}^2, d_T)$ with the taxicab metric

$$d_T((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

(5) The open ball $B((3, 2), 1)$ in the metric space $(\mathbb{R}^2, d)$ with the discrete metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}.$$

What is the difference between $B((3, 2), 1)$ and $B((3, 2), r)$ in this metric space if $r > 1$? If $r < 1$?

Neighborhoods

We are familiar with the idea of open intervals in $\mathbb{R}$. We next introduce the idea of an open neighborhood of a point and characterize continuity in terms of neighborhoods. This is the next step in developing the notation of continuity in topological spaces.

The open ball $B(a,\delta)$ in a metric space (X,d) is also called the δ-*neighborhood* around a. A neighborhood of a point can be thought of as any set that envelops that point.

Definition 7.2. Let (X,d_X) be a metric space, and let $a \in X$. A subset N of X is a **neighborhood** of a if there exists a $\delta > 0$ such that $B(a,\delta) \subseteq N$.

Example 7.3.

- In $\mathbb{R}$ with the Euclidean metric, the set $\mathbb{R}^+$ (the positive real numbers) is a neighborhood of $a = 1$ because the open ball $B(1, 0.5)$ is completely contained in $\mathbb{R}^+$.
- In $\mathbb{R}$ with the Euclidean metric, the set $\mathbb{Z}$ is not a neighborhood of $a = 1$ because any open ball centered at $a = 1$ will contain some non-integers.
- In $\mathbb{R}$ with the discrete metric, the set $\mathbb{Z}$ is a neighborhood of $a = 1$ because the open ball $B(a, 1) = \{a\}$.

As another example, the open ball $B(a,\delta)$ is a neighborhood of a. We can say even more about open balls.

Activity 7.1. Let (X,d) be a metric space, let $a \in X$, and let $\delta > 0$. In this activity we ask the question, is $B(a,\delta)$ a neighborhood of each of its points?

(a) Let $b \in B(a,\delta)$. What do we have to do to show that $B(a,\delta)$ is a neighborhood of b?

(b) Use Figure 7.1 to help show that $B(a,\delta)$ is a neighborhood of b.

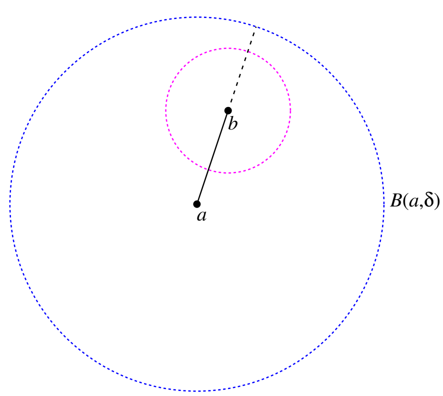

Figure 7.1: $B(a,\delta)$ as a neighborhood of b.

(c) Is the converse true? That is, if a set is a neighborhood of each of its points, is the set an open ball? No proof is necessary, but a convincing argument is in order.

Continuity and Neighborhoods

We can define continuity now in terms of neighborhoods instead of using metrics. The advantage here is that this idea does not explicitly depend on the existence of a metric, so we will be able to adopt this concept of continuity for arbitrary topological spaces.

Recall that a function f from a metric space (X, d_X) to a metric space (Y, d_Y) is continuous at $a \in X$ if, for any $\epsilon > 0$ there exists $\delta > 0$ so that $d_X(x, a) < \delta$ implies $d_Y(f(x), f(a)) < \epsilon$. We can interpret this definition of continuity to say that for every $\epsilon > 0$, the inverse image under f of the open ball $B(f(a), \epsilon)$ contains the open ball $B(a, \delta)$ for some $\delta > 0$. It is not unreasonable to wonder if the set $f^{-1}(B(f(a), \epsilon))$ itself is an open ball. We investigate this question in the following activity.

Activity 7.2. Let f be a function from a metric space (X, d_X) to a metric space (Y, d_Y) that is continuous at $a \in X$. Using the notation from the paragraph above, in this activity we determine if $f^{-1}(B(f(a), \epsilon))$ must equal $B(a, \delta)$ for some δ.

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^2,$$

where we use the Euclidean metric d_E throughout. Assume that f is a continuous function. Then f is continuous at $x = 2$.

(a) What is $B(f(2), 1)$?

(b) What is $f^{-1}(B(f(2), 1))$?

(c) Is $f^{-1}(B(f(2), 1))$ an open ball centered at 2? Explain.

The conclusion to be drawn from Activity 7.2 is that if f is continuous, we can only conclude that the inverse image of $B(f(a), \epsilon)$ *contains* an open ball centered at a. By definition of continuity, if for every $\epsilon > 0$ there exists a $\delta > 0$ so that the open ball $f^{-1}(B(f(a), \epsilon))$ contains $B(a, \delta)$, then f is continuous at a. We summarize this in the next theorem.

Theorem 7.4. *Let f be a function a metric space (X, d_X) to a metric space (Y, d_Y), and let $a \in X$. Then f is continuous at $a \in X$ if and only if, given any $\epsilon > 0$ there exists $\delta > 0$ so that*

$$B(a, \delta) \subseteq f^{-1}(B(f(a), \epsilon)).$$

We can extend this idea of continuity to describe continuity in terms of neighborhoods. This condition will allow us to later consider continuous functions even if there are no metrics on our spaces.

Theorem 7.5. *Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. Then f is continuous at $a \in X$ if and only if the inverse image of every neighborhood of $f(a)$ is a neighborhood of a.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. To prove this biconditional statement we need to prove both implications. First assume that f is continuous at some point $a \in X$. We will show that for any neighborhood N of $f(a)$ in Y, its inverse image

$f^{-1}(N)$ in X is a neighborhood of a in X. Let N be a neighborhood of $f(a)$ in Y. To demonstrate that $f^{-1}(N)$ is a neighborhood of a in X, we need to find an open ball around a that is contained in $f^{-1}(N)$. Since N is a neighborhood of $f(a)$, by definition there exists $\epsilon > 0$ so that $B(f(a), \epsilon) \subseteq N$. Since f is continuous at a, there exists $\delta > 0$ such that $B(a, \delta) \subseteq f^{-1}(B(f(a), \epsilon))$. So if $x \in B(a, \delta)$, then $f(x) \in B(f(a), \epsilon) \subseteq N$. So $B(a, \delta) \subseteq f^{-1}(N)$, and $f^{-1}(N)$ is a neighborhood of a in X.

The proof of the reverse implication is left for the next activity. ■

Activity 7.3. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. Let $a \in X$. In this activity we prove that if the inverse image of every neighborhood of $f(a)$ is a neighborhood of a, then f is continuous at a.

(a) What does Theorem 7.4 tell us that we need to do to show that f is continuous at a?

(b) Suppose ϵ is greater than 0, why is $B(f(a), \epsilon)$ a neighborhood of $f(a)$ in Y?

(c) What does our hypothesis tell us about $f^{-1}(B(f(a), \epsilon))$?

(d) What can we conclude from part (c)?

(e) How do (a)-(d) show that f is continuous at a?

We conclude this section with some important facts about neighborhoods. Assume that (X, d) is a metric space and $a \in X$.

- There is a neighborhood that contains a.
- If N is a neighborhood of a and $N \subseteq M$, then M is a neighborhood of a.
- If M and N are neighborhoods of a, then so is $M \cap N$.

The proofs are straightforward and left for Exercise (8).

Summary

Important ideas that we discussed in this section include the following.

- If (X, d) is a metric space and $a \in X$, then an open ball centered at a is a set of the form
$$B(a, \delta) = \{x \in X \mid d(x, a) < \delta\}$$
for some positive number δ.
- A subset N of a metric space (X, d) is s neighborhood of a point $a \in N$ if there is a positive real number δ such that $B(a, \delta) \subseteq N$.
- An important property of open balls is that every open ball is a neighborhood of each of its points. This is our first step toward defining the concept of open sets that will form the foundation for topological spaces.
- A function f from a metric space (X, d_X) to a metric space (Y, d_Y) is continuous at $a \in X$ if $f^{-1}(N)$ is a neighborhood of a in X for any neighborhood N of $f(a)$ in Y.

Exercises

(1) Determine, with proof, which of the following sets A is a neighborhood of a in the indicated metric space.

(a) $A = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$ in $(\mathbb{R}^2, d_E)$ with $a = (0.5, 0.5)$

(b) A is the x-axis in $(\mathbb{R}^2, d_T)$ with $a = (0, 0)$, where d_T is the taxicab metric

(c) A is the set of rational numbers in $(\mathbb{R}, d_E)$ with $a = 0$

(d) A is the set of positive integers in (Q, d) and $a = 1$, where Q is the set of all rational numbers in reduced form with metric $d : Q \times Q \to \mathbb{R}$ defined by

$$d\left(\frac{a}{b}, \frac{c}{d}\right) = \max\{|a - c|, |b - d|\}$$

(The fact that d is a metric is the topic of Exercise 3 on page 40.)

(2) Let $X = \{1, 3, 5\}$ and define $d_X : X \times X \to \mathbb{R}$ by $d_X(x, y) = xy - 1 \pmod 8$. That is, $d_X(x, y)$ is the remainder when $xy - 1$ is divided by 8. That d_X is a metric on X is examined in Exercise 2 on page 40. Let (Y, d_Y) be a metric space. Is it possible to define a function $f : X \to Y$ that is not continuous? Explain.

(3) If $x = (x_1, x_2, \ldots, x_n)$, we let $|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$. For $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots y_n)$, define $d_H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$d_H(x, y) = \begin{cases} 0 & \text{if } x = y \\ |x| + |y| & \text{otherwise.} \end{cases}$$

The fact that d_H is a metric is examined in Exercise 7 on page 41.

Let $(X, d_X) = (\mathbb{R}^2, d_H)$ and let $(Y, d_Y) = (\mathbb{R}, d_E)$. Define $f : X \to Y$ and $g : X \to Y$ by

$$f(x) = \begin{cases} 0 & \text{if } x = (0, 0) \\ 1 & \text{otherwise} \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0 & \text{if } |x| < 1 \\ 1 & \text{otherwise} \end{cases}.$$

One of f, g is continuous and the other is not. Determine which is which, with proof for each.

(4) Recall from Section 3 that we can construct a finite metric space by starting with a finite set of points and making a graph with the points as vertices. We construct edges so that the graph is connected (that is, there is a path from any one vertex to any other) and give weights to the edges. We then define a metric d on S by letting $d(x, y)$ be the length of a shortest path between vertices x and y in the graph.

Consider the metric space (X, d) corresponding to the graph in Figure 7.2.

(a) Determine all of the open balls $B(a, \delta)$ for every positive real number δ.

(b) Find all of the neighborhoods of a.

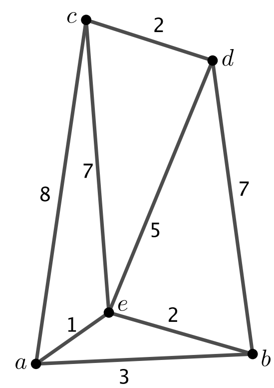

Figure 7.2: A graph to define a metric.

(5) (a) Let $f : (\mathbb{R}, d_E) \to (\mathbb{R}, d_E)$ be defined by $f(x) = ax + b$ for some real numbers a and b with $a \neq 0$. Let $p \in \mathbb{R}$ and let $r > 0$. Show that $f^{-1}(B(f(p), r))$ contains an open ball centered at p. Conclude that every linear function from $(\mathbb{R}, d_E)$ to $(\mathbb{R}, d_E)$ is continuous. (Hint: By Exercise 5 on page 63 we can assume $a > 0$ to simplify the problem.)

(b) Let $f : (\mathbb{R}, d_E) \to (\mathbb{R}, d_E)$ be defined by $f(x) = ax^2 + bx + c$ for some real numbers a, b, and c with $a \neq 0$. Let $p \in \mathbb{R}$ and let $r > 0$. $f^{-1}(B(f(p), r))$ contains an open ball centered at p. Conclude that every quadratic function from $(\mathbb{R}, d_E)$ to $(\mathbb{R}, d_E)$ is continuous. (Hint: Consider cases.)

(6) Let (X, d) be a metric space, and let A be a nonempty subset of X. Exercise 11 on page 55 tells us that

$$d(b, A) \leq d(b, c) + d(c, A)$$

for all $b, c \in X$.

Define $f : X \to \mathbb{R}$ by $f(x) = d(x, A)$. Let $b \in X$. Given $\epsilon > 0$, show that there is a neighborhood N of b such that $x \in N$ implies $f(x) \in B(f(b), \epsilon)$. Conclude that f is a continuous function. (Assume the metric on $\mathbb{R}$ is the Euclidean metric.)

(7) Let a and b be distinct points of a metric space X. Prove that there are neighborhoods N_a and N_b of a and b respectively such that $N_a \cap N_b = \emptyset$.

(8) Let (X, d) be a metric space and let $a \in X$. Prove each of the following.

(a) There is a neighborhood that contains a.

(b) If N is a neighborhood of a and $N \subseteq M$, then M is a neighborhood of a.

(c) If M and N are neighborhoods of a, then so is $M \cap N$.

(9) Let $f : (\mathbb{R}, d_E) \to (\mathbb{R}, d_E)$ be a continuous function. Show that if $f(a) > 0$ for some $a \in \mathbb{R}$, then there is a neighborhood N of a such that $f(x) > 0$ for all $x \in N$.

(10) Let (X, d) be a metric space where d is the discrete metric. Show that every subset of X is a neighborhood of each of its points.

(11) For each of the following, answer true if the statement is always true. If the statement is only sometimes true or never true, answer false and provide a concrete example to illustrate that the statement is false. If a statement is true, explain why.

(a) If N is a neighborhood of a point a in a metric space X, then any open ball contained in N is also a neighborhood of a.

(b) If N is a neighborhood of a point a in a metric space X, then N is a neighborhood of each of its points.

(c) If X and Y are metric spaces and $f : X \to Y$ is a continuous function, then $f(N)$ is a neighborhood of $f(a)$ in Y whenever N is a neighborhood of a in X.

(d) If X and Y are metric spaces and $f : X \to Y$ is continuous at $a \in X$, and N is a neighborhood of $f(a)$ in Y, then $f^{-1}(N)$ is a neighborhood of a in X.

(e) If a is a point in a metric space X and if δ is a positive real number, then the open ball $B(a, \delta)$ contains infinitely many points of X.

(f) If N_1, N_2, $\ldots$, N_k are neighborhoods of a point a in a metric space X for some positive integer k, then $\bigcap_{i=1}^{k} N_i$ is a neighborhood of a.

(g) If N_α is a neighborhood of a point a in a metric space X for all α in some indexing set I, then $\bigcap_{\alpha \in I} N_\alpha$ is a neighborhood of a.