Section 8

Open Sets in Metric Spaces

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is an open set in a metric space?
- What is an interior point of a subset of a metric space? How are interior points related to open sets?
- What is the interior of a set? How is the interior of a set related to open sets?
- How can we use open sets to determine the continuity of a function?
- What important properties do open sets have in relation to unions and intersections?

Introduction

Consider the interval (a, b) in $\mathbb{R}$ using the Euclidean metric. If $m = \frac{a+b}{2}$, then $(a, b) = B\left(m, \frac{b-a}{2}\right)$, so every open interval is an open ball. As an open ball, an open interval (a, b) is a neighborhood of each of its points. This is the foundation for the definition of an open set in a metric space.

Recall that we defined a subset N of X to be neighborhood of point a in a metric space (X, d) if N contains an open ball $B(a, \epsilon)$ for some $\epsilon > 0$. We saw that every open ball is a neighborhood of each of its points, and we will now extend that idea to define an *open set* in a metric space.

Definition 8.1. A subset O of a metric space X is an **open set** if O is a neighborhood of each of its points.

So, by definition, any open ball is an open set. Also by definition, open sets are neighborhoods of each of their points. Open sets are different than non-open sets. For example, $(0, 1)$ is an open set in $\mathbb{R}$ using the Euclidean metric, but $[0, 1)$ is not. The reason $[0, 1)$ is not an open set is that there is no open ball centered at 0 that is entirely contained in $[0, 1)$. So 0 has a different property than the other points in $[0, 1)$. The set $[0, 1)$ is a neighborhood of each of the points in $(0, 1)$, but is not a neighborhood of 0. We can think of the points in $(0, 1)$ as being in the interior of the set $[0, 1)$. This leads to the next definition.

Definition 8.2. Let A be a subset of a metric space X. A point $a \in A$ is an **interior point** of A if A is a neighborhood of a.

As we will soon see, open sets can be characterized in terms of interior points.

Preview Activity 8.1.

(1) Determine if the set A is an open set in the metric space (X, d). Explain your reasoning.

(a) $X = \mathbb{R}$, $d = d_E$, the Euclidean metric, $A = [0, 0.5)$.

(b) $X = \{x \in \mathbb{R} \mid 0 \leq x \leq 1\}$, $d = d_E$, the Euclidean metric, $A = [0, 0.5)$. Assume that the Euclidean metric is a metric on X.

(c) $X = \{a, b, c, d\}$, d is the discrete metric defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y, \end{cases}$$

and $A = \{a, b\}$.

(2)

(a) What are the interior points of the following sets in $(\mathbb{R}, d_E)$? Explain.

$$(0, 1) \quad (0, 1] \quad [0, 1) \quad [0, 1].$$

(b) Let $A = \{0, 1, 2\}$ in $(\mathbb{R}, d_E)$. What are the interior points of A? Explain.

(c) Let $\mathbb{Q}$ be the set of rational numbers in $(\mathbb{R}, d_E)$. What are the interior points of $\mathbb{Q}$? Explain.

Open Sets

Open sets are vitally important in topology. In fact, we will see later that every topological space is completely defined by its open sets. Recall that an open ball is an open set. There are other subsets that every metric space contains, and we might ask if they are open or not.

Activity 8.1. Let X be a metric space.

(a) Is $\emptyset$ an open set in X? Explain.

(b) Is X an open set in X? Explain.

We have defined open balls, and open balls are the canonical examples of open sets. In fact, as the following theorem shows, the open balls determine the open sets.

Theorem 8.3. *Let X be a metric space. A subset O of X is open if and only if O is a union of open balls.*

Proof. Let X be a metric space and O a subset of X. To prove this biconditional statement we first assume that O is an open set and demonstrate that O is a union of open balls. Let $a \in O$. Since O is open, there exists $\epsilon_a > 0$ so that $B(a, \epsilon_a) \subseteq O$. We will show that

$$O = \bigcup_{a \in O} B(a, \epsilon_a).$$

By the way we chose ϵ_a, $B(a, \epsilon_a) \subseteq O$ for every $a \in O$. So $\bigcup_{a \in O} B(a, \epsilon_a) \subseteq O$. For the reverse containment, let $x \in O$. Then $x \in B(x, \epsilon_x)$ and so $x \in \bigcup_{a \in O} B(a, \epsilon_a)$. Thus, $O \subseteq \bigcup_{a \in O} B(a, \epsilon_a)$. We conclude that O is a union of open balls if O is an open set.

The proof of the converse is left for the following activity. ∎

Activity 8.2. Let X be a metric space. To prove the remaining implication of Theorem 8.3, assume that a subset O of X is a union of open balls.

(a) What do we need to show to prove that O is an open set?

(b) Let $x \in O$. Why is there an open ball B in O that contains x?

(c) Complete the proof to show that O is an open set.

Theorem 8.3 tells us that every open set is made up of open balls, so the open balls generate all open sets much like a basis of a vector space in linear algebra generates all of the elements of the vector space. For this reason we call the set of open balls in a metric space a *basis* for the open sets of the metric space. We will discuss this idea in more detail in a subsequent section.

Unions and Intersections of Open Sets

Once we have defined open sets we might wonder about what happens if we take a union or intersection of open sets.

Activity 8.3.

(a) Let $A = (-2, 1)$ and $B = (-1, 2)$ in $(\mathbb{R}, d_E)$.

  i. Is $A \cup B$ open? Explain.

  ii. Is $A \cap B$ open? Explain.

(b) Let $X = \mathbb{R}$ with the Euclidean metric. Let $A_n = \left(1 - \frac{1}{n}, 1 + \frac{1}{n}\right)$ for each $n \in \mathbb{Z}^+$.

  i. What is $\bigcup_{n \geq 1} A_n$? A proof is not necessary.

  ii. Is $\bigcup_{n \geq 1} A_n$ open in $\mathbb{R}$? Explain.

  iii. What is $\bigcap_{n \geq 1} A_n$? A proof is not necessary.

  iv. Is $\bigcap_{n \geq 1} A_n$ open in $\mathbb{R}$? Explain.

Activity 8.3 demonstrates that an arbitrary intersection of open sets is not necessarily open. However, there are some things we can say about unions and intersections of open sets.

Theorem 8.4. *Let X be a metric space.*

(1) Any union of open sets in X is an open set in X.

(2) Any finite intersection of open sets in X is an open set in X.

Proof. Let X be a metric space. To prove part 1, assume that $\{O_\alpha\}$ is a collection of open sets in X for α in some indexing set I and let $O = \bigcup_{\alpha \in I} O_\alpha$. By Theorem 8.3, we know that O_α is a union of open balls for each $\alpha \in I$. Combining all of these open balls together shows that O is a union of open balls and is therefore an open set by Theorem 8.3.

For part 2, assume that O_1, O_2, ..., O_n are open sets in X for some $n \in \mathbb{Z}^+$. To show that $O = \bigcap_{k=1}^n O_k$ is an open set, we will show that O is a neighborhood of each of its points. Let $x \in O$. Then $x \in O_k$ for each $1 \leq k \leq n$. Let k be between 1 and n. Since O_k is open, we know that O_k is a neighborhood of each of its points. So there exists $\epsilon_k > 0$ such that $B(x, \epsilon_k) \subseteq O_k$. Since there are only finitely many values of k, let $\epsilon = \min\{\epsilon_k \mid 1 \leq k \leq n\}$. Then $B(x, \epsilon) \subseteq B(x, \epsilon_k)$ for each k and so $B(x, \epsilon) \subseteq \bigcap_{k=1}^n O_k = O$. Therefore, O is a neighborhood of each of its points and O is an open set. ■

Continuity and Open Sets

Recall that we showed that a function f from a metric space (X, d_X) to a metric space (Y, d_Y) is continuous if and only if $f^{-1}(N)$ is a neighborhood of $a \in X$ whenever N is a neighborhood of $f(a)$ in Y. We can now provide another characterization of continuous functions in terms of open sets. This is the characterization that will serve as our definition of continuity in topological spaces.

Theorem 8.5. *Let f be a function from a metric space (X, d_X) to a metric space (Y, d_Y). Then f is continuous if and only if $f^{-1}(O)$ is an open set in X whenever O is an open set in Y.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function. To prove this biconditional statement we need to prove both implications. First assume that f is a continuous function. We must show that $f^{-1}(O)$ is an open set in X for every open set O in Y. So let O be an open set in Y. To demonstrate that $f^{-1}(O)$ is open in X, we will show that $f^{-1}(O)$ is a neighborhood of each of its points. Let $a \in f^{-1}(O)$. Then $f(a) \in O$. Now O is an open set, so there is an open ball $B(f(a), \epsilon)$ around $f(a)$ that is entirely contained in O. Since $B(f(a), \epsilon)$ is a neighborhood of $f(a)$, we know that $f^{-1}(B(f(a), \epsilon))$ is a neighborhood of a. Thus, there exists $\delta > 0$ so that $B(a, \delta) \subseteq f^{-1}(B(f(a), \epsilon))$. Now $f(B(a, \delta)) \subseteq B(f(a), \epsilon) \subseteq O$, and so $B(a, \delta) \subseteq f^{-1}(O)$. We conclude that $f^{-1}(O)$ is a neighborhood of each of its points and is therefore an open set in X.

The proof of the reverse implication is left for the next activity. ■

Activity 8.4. Let f be a function from a metric space (X, d_X) to a metric space (Y, d_Y).

(a) What assumption do we make to prove the remaining implication of Theorem 8.5? What do we need to demonstrate to prove the conclusion?

(b) Let $a \in X$, and let N be a neighborhood of $f(a)$ in Y. Why does there exist an $\epsilon > 0$ so that $B(f(a), \epsilon) \subseteq N$.

(c) What does our hypothesis tell us about $f^{-1}(B(f(a), \epsilon))$ in X?

(d) Why is $f^{-1}(N)$ a neighborhood of a? How does this show that f is a continuous function?

Recall that every open set is a union of open balls, so we can simplify proofs of continuous functions in metric spaces by working only with open balls instead of arbitrary open sets. The next activity provides the details.

Activity 8.5. In this activity we prove the following corollary to Theorem 8.5.

Corollary 8.6. *A function f from a metric space (X, d_X) to a metric space (Y, d_Y) is continuous if and only if $f^{-1}(B)$ is open in X whenever B is an open ball in Y.*

To set up the proof, let (X, d_X) and (Y, d_Y) be metric spaces, and let $f : X \to Y$ be a function.

(a) Since the corollary is a biconditional statement, we need to prove both implications. First, assume that f is continuous. Use Theorem 8.5 to explain why $f^{-1}(B)$ is open in X whenever B is an open ball in Y.

(b) For the remaining implication, assume that $f^{-1}(B)$ is an open set in X for any open ball B in Y. To show that f is a continuous function, we will use Theorem 8.5 and show that $f^{-1}(O)$ is open in X whenever O is an open set in Y. So let O be an open set in Y.

i. What does Theorem 8.3 tell us about O.

ii. Recall that Lemma 2.11 tells us that if $\{B_\beta\}$ is a collection of subsets of Y for β in some indexing set J, then

$$f^{-1}\left(\bigcup_{\beta \in J} B_\beta\right) = \bigcup_{\beta \in J} f^{-1}(B_\beta).$$

Use Lemma 2.11 to show that $f^{-1}(O)$ is open in X and conclude that f is a continuous function.

Example 8.7. As an example of Corollary 8.6, we prove that the square function from $\mathbb{R}$ to $\mathbb{R}$ is a continuous function. Let $X = \mathbb{R}$ with the Euclidean metric d_E, and let $f : X \to X$ be defined by $f(x) = x^2$. We will show that f is a continuous function by verifying that $f^{-1}(B)$ is open in X for every open ball B in X. Let $B = B(b, \beta) = (b - \beta, b + \beta)$ be an open ball in X. Let $B' = B(b, \beta) \cap (\mathbb{R}^+ \cup \{0\})$. We consider cases.

- Suppose that $B' = \emptyset$. Then $f^{-1}(B) = \emptyset$ and $f^{-1}(B)$ is open in X.
- Suppose that $B' = [0, b + \beta)$. Then $f^{-1}(B) = (-\sqrt{b+\beta}, \sqrt{b+\beta})$ and $f^{-1}(B)$ is open in X.
- The final case is $B' = (b - \beta, b + \beta)$. Then

$$f^{-1}(B) = (-\sqrt{b+\beta}, -\sqrt{b-\beta}) \cup (\sqrt{b-\beta}, \sqrt{b+\beta})$$

and $f^{-1}(B)$ is open in X.

Since the inverse image of every open ball is an open set, we conclude that f is a continuous function.

The Interior of a Set

Open sets can be characterized in terms of their interior points. By definition, every open set is a neighborhood of each of its points, so every point of an open set O is an interior point of O. Conversely, if every point of a set O is an interior point, then O is a neighborhood of each of its points and is open. This argument is summarized in the next theorem.

Theorem 8.8. *Let X be a metric space. A subset O of X is open if and only if every point of O is an interior point of O.*

The collection of interior points in a set form a subset of that set, called the *interior* of the set.

Definition 8.9. The **interior** of a subset A of a metric space X is the set

$$\text{Int}(A) = \{a \in A \mid a \text{ is an interior point of } A\}.$$

Activity 8.6. Determine $\text{Int}(A)$ for each of the sets A.

(a) $A = (0, 1]$ in $(\mathbb{R}, d_E)$

(b) $A = [0, 1]$ in $(\mathbb{R}, d_E)$

(c) $A = \{-2\} \cup [0, 5] \cup \{7, 8, 9\}$ in $(\mathbb{R}, d_E)$

One might expect that the interior of a set is an open set. This is true, but we can say even more. As Theorem 8.10 will show, if A is a subset of a metric space X, not only is $\text{Int}(A)$ an open set, but every open set that is contained in A is a subset of $\text{Int}(A)$. So $\text{Int}(A)$ is the largest, in the sense of containment, open subset of X that contains A.

Theorem 8.10. *Let (X, d) be a metric space, and let A be a subset of X. Then interior of A is the largest open subset of X contained in A.*

Proof. Let (X, d) be a metric space, and let A be a subset of X. We need to prove that $\text{Int}(A)$ is an open set in X, and that $\text{Int}(A)$ is the largest open subset of X contained in A. First we demonstrate that $\text{Int}(A)$ is an open set. Let $a \in \text{Int}(A)$. Then a is an interior point of A, so A is a neighborhood of a. This implies that there exists an $\epsilon > 0$ so that $B(a, \epsilon) \subseteq A$. But $B(a, \epsilon)$ is a neighborhood of each of its points, so every point in $B(a, \epsilon)$ is an interior point of A. It follows that $B(a, \epsilon) \subseteq \text{Int}(A)$. Thus, $\text{Int}(A)$ is a neighborhood of each of its points and, consequently, $\text{Int}(A)$ is an open set.

The proof that $\text{Int}(A)$ is the largest open subset of X contained in A is left for the next activity. ∎

Activity 8.7. Let (X, d) be a metric space, and let A be a subset of X.

(a) What will we have to show to prove that $\text{Int}(A)$ is the largest open subset of X contained in A?

(b) Suppose that O is an open subset of X that is contained in A, and let $x \in O$. What does the fact that O is open tell us?

(c) Complete the proof that $O \subseteq \text{Int}(A)$.

One consequence of Theorem 8.10 is the following.

Corollary 8.11. *A subset O of a metric space X is open if and only if $O = \text{Int}(O)$.*

The proof is left for Exercise (2).

Summary

Important ideas that we discussed in this section include the following.

- A subset O of a metric space (X, d) is an open set if O is a neighborhood of each of its points. Alternatively, O is open if O is a union of open balls.
- A point a in a subset A of a metric space (X, d) is an interior point of A if A is a neighborhood of a. A set O is open if every point of O is an interior point of O.
- The interior of a set is the set of all interior points of the set. The interior of a set A in a metric space X is the largest open subset of X contained in A. A set is open if and only if the set is equal to its interior.
- A function f from a metric space X to a metric space Y is continuous if $f^{-1}(O)$ is open in X whenever O is open in Y.
- Any union of open sets is open, while any finite intersection of open sets is open.

Exercises

(1) Let d be the discrete metric. Let (X, d) be a metric space.

  (a) Show that every subset of X is open.

  (b) Let (Y, d_Y) be a metric space. Prove that every function $f : X \to Y$ is continuous.

  (c) Is it also true that every function $f : Y \to X$ is continuous? If yes, prove your answer. If no, find a counterexample.

(2) Prove that a subset O of a metric space X is open if and only if $O = \text{Int}(O)$.

(3) Let A and B be subsets of a metric space X with $A \subseteq B$. Prove or disprove the following.

  (a) $\text{Int}(A) \subseteq \text{Int}(B)$

  (b) $\text{Int}(\text{Int}(A)) = \text{Int}(A)$

(4) Let $a, b \in \mathbb{R}$ with $a < b$. Show that the set $(a, b]$ in $(\mathbb{R}, d_E)$ is not an open set.

(5) Let $A = \{(x, y) \in \mathbb{R}^2 \mid 1 < x < 3, 0 < y < 1\}$.

  (a) Is A an open set in $(\mathbb{R}^2, d_E)$? Prove your answer.

  (b) Is A an open set in $(\mathbb{R}^2, d_T)$? Prove your answer.

  (c) Is A an open set in $(\mathbb{R}^2, d_M)$? Prove your answer.

(6) Let S be a finite set of points in $\mathbb{R}^2$. Is the set $\mathbb{R}^2 \setminus A$ an open set in $(\mathbb{R}^2, d_E)$? Prove your answer.

(7) Let (X, d_X) and (Y, d_Y) be metric spaces and let $f : X \to Y$ be a function. Prove that f is continuous if and only if $f^{-1}(\text{Int}(B)) \subseteq \text{Int}(f^{-1}(B))$ for every subset B of Y.

(8) Consider the metric space (Q, d), where $d : Q \times Q \to \mathbb{R}$ is defined by

$$d\left(\frac{a}{b}, \frac{u}{v}\right) = \max\{|a - u|, |b - v|\}$$

(The fact that d is a metric is the topic of Exercise 3 on 40.) Describe the open ball $B(q, 2)$ in Q if $q = \frac{2}{5}$.

(9) Let (X, d) be a metric space and let x_1 and x_2 be distinct points in X. Prove that there are open sets O_1 containing x_1 and O_2 containing x_2 such that $O_1 \cap O_2 = \emptyset$. (This shows that we can separate points in metric spaces with open sets. Separation properties are important in topology.)

(10) Let $X = \mathbb{R}$ with the Euclidean metric d_E, and let $Y = \mathbb{R}$ with the metric d defined by $d(x, y) = \frac{|x-y|}{|x-y|+1}$ (that d is a metric is the subject of Exercise (11) on 42). Let a and b be real numbers and let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = ax + b$. That is, f is an arbitrary linear function from $\mathbb{R}$ to $\mathbb{R}$.

  (a) Describe the open balls in (Y, d). That is, if a is a real number and δ is a positive real number, what is the specific set of points in $B(a, \delta)$ in Y?

  (b) Is f from X to Y continuous for any real numbers a and b? Prove your answer.

  (c) Is f from Y to X continuous for any real numbers a and b? Prove your answer.

(11) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by $f((x, y)) = x$. Assume that we use the max metric d_M on $\mathbb{R}^2$ and the Euclidean metric d_E on $\mathbb{R}$. Use Theorem to determine if f a continuous function. Prove your conjecture.

(12) For each of the following, answer true if the statement is always true. If the statement is only sometimes true or never true, answer false and provide a concrete example to illustrate that the statement is false. If a statement is true, explain why.

  (a) If A and B are nonempty subsets of a metric space X, then $\text{Int}(A \cup B) \subseteq \text{Int}(A) \cup \text{Int}(B)$.

  (b) If A and B are nonempty subsets of a metric space X, then $\text{Int}(A) \cup \text{Int}(B) \subseteq \text{Int}(A \cup B)$.

(c) If A and B are nonempty subsets of a metric space X, then Int$(A \cap B) \subseteq$ Int$(A) \cap$ Int(B).

(d) If A and B are nonempty subsets of a metric space X, then Int$(A) \cap$ Int$(B) \subseteq$ Int$(A \cap B)$.

(e) Every subset of an open set in a metric space (X, d) is open in X.

(f) A subset O of $\mathbb{R}^2$ is open under the Euclidean metric d_E if and only if O is open under the taxicab metric d_T.

(g) Let X = $[0, 1] \cup [2, 3]$ endowed with the Euclidean metric. Then $[0, 1]$ is an open subset of X.