Appendix A

Complex Numbers

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is a complex number?
- How is the sum and product of two complex numbers defined?
- How do we find the multiplicative inverse of a nonzero complex number?
- What general structure does the set of complex numbers have?

Complex Numbers

Complex numbers are usually introduced as a tool to solve the quadratic equation $x^2+1=0$. However, that is not how complex numbers first came to light. The story actually involves solutions to the general cubic equation. The interested reader could consult Chapter 6 of William Dunham's excellent book *Journey Through Genius*. In this appendix we touch on the basics of complex numbers to provide enough context for the section on complex eigenvalues.

A complex number is defined by a pair of real numbers - the *real part* of the complex number and the *imaginary part* of the complex number.

Definition A.1. A **complex number** is a number of the form

$$a+bi$$

where a and b are real numbers and $i^2=-1$.

The number a is the real part of the complex number and the number b is the imaginary part. We often write

$$z=a+bi$$

for a complex number z. We say that two complex numbers $a + bi$ and $c + di$ are equal if $a = c$ and $b = d$.

There is an arithmetic of complex numbers that is determined by an addition and multiplication of complex numbers. Adding complex numbers is natural:

$$(a + bi) + (c + di) = (a + c) + (b + d)i.$$

That is, to add two complex numbers we add their real parts together and their imaginary parts together.

Activity A.1. Multiplication of complex numbers is is also done in a natural way.

(a) By expanding the product as usual, treating i as we would any real number, and exploiting the fact that $i^2 = -1$, explain why we define the product of complex numbers $a + bi$ and $c + di$ as

$$(a + bi)(c + di) = (ac - bd) + (bc + ad)i.$$

(b) Use the definitions of addition and multiplication to write each of the sums or products as a complex number in the form $a + bi$.

i. $(2 + 3i) + (7 - 4i)$
ii. $(4 - 2i)(3 + i)$
iii. $(2 + i)i - (3 + 4i)$

It isn't difficult to show that the set of complex numbers, which we denote by $\mathbb{C}$, satisfies many useful and familiar properties.

Activity A.2. Show that $\mathbb{C}$ has the same structure as $\mathbb{R}$. That is, show that for all u, w, and z in $\mathbb{C}$, the following properties are satisfied.

(a) $w + z \in \mathbb{C}$ and $wz \in \mathbb{C}$

(b) $w + z = z + w$ and $wz = zw$

(c) $(w + z) + u = w + (z + u)$ and $(wz)u = w(zu)$

(d) There is an element 0 in $\mathbb{C}$ such that $z + 0 = z$

(e) There is an element 1 in $\mathbb{C}$ such that $(1)z = z$

(f) There is an element $-z$ in $\mathbb{C}$ such that $z + (-z) = 0$

(g) If $z \neq 0$, there is an element $\frac{1}{z}$ in $\mathbb{C}$ such that $z\left(\frac{1}{z}\right) = 1$

(h) $u(w + z) = (uw) + (uz)$

The result of Activity A.2 is that, just like $\mathbb{R}$, the set $\mathbb{C}$ is a field. If we wanted to, we could define vector spaces over $\mathbb{C}$ just like we did over $\mathbb{R}$. The same results hold.

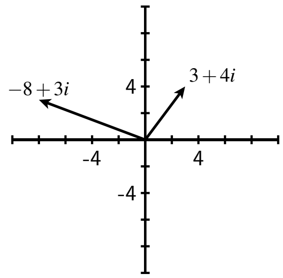

Figure A.1: Two complex numbers.

Conjugates and Modulus

We can draw pictures of complex numbers in the plane. We let the x-axis be the real axis for a complex number and the y-axis the imaginary axis. That is, if $z = a + bi$ we can think of z as a directed line segment from the origin to the point (a, b), where the terminal point of the segment is a units from the imaginary axis and b units from the real axis. For example, the complex numbers $3 + 4i$ and $-8 + 3i$ are shown in Figure A.1.

We can also think of the complex number $z = a + bi$ as the vector $[a\ b]^{\mathsf{T}}$. In this way, the set $\mathbb{C}$ is a two-dimensional vector space over $\mathbb{R}$ with basis $\{1, i\}$. Each of these complex numbers has a length that we call the *norm* or *modulus* of the complex number. We denote the norm of a complex number $a + bi$ as $|a + bi|$. The distance formula or the Pythagorean theorem show that

$$|a + bi| = \sqrt{a^2 + b^2}.$$

Note that

$$a^2 + b^2 = a^2 - b^2 i^2 = (a + bi)(a - bi)$$

so the norm of the complex number $a + bi$ can also be viewed as a square root of the product of $a+bi$ with $a-bi$. The number $a-bi$ is called the *complex conjugate* of $a+bi$. If we let $z = a+bi$, we denote the complex conjugate of z as $\overline{z}$. So $\overline{a + bi} = a - bi$.

Activity A.3. Let $w = 2 + 3i$ and $z = -1 + 5i$.

(a) Find $\overline{w}$ and $\overline{z}$.

(b) Compute $|w|$ and $|z|$.

(c) Compute $w\overline{w}$ and $z\overline{z}$.

(d) Let z be an arbitrary complex number. There is a relationship between $|z|$, z, and $\overline{z}$. Find and verify this relationship.

(e) What is $\overline{z}$ if $z \in \mathbb{R}$?