Appendix E

The Cubic Formula

The quadratic formula tells us how to find all solutions to quadratic equations in $\mathbb{C}[x]$. There is also a general formula for solving cubic equations, although it is much more complicated than the quadratic formula.

First note that when we want to find roots of polynomials, it suffices to work with only monic polynomials. To see why, consider the general polynomial $f(x) = \sum_{i=0}^{n} a_i x^i$ (with $a_n \neq 0$) in $F[x]$, where F is a field. Let $g(x) = x^n + \sum_{i=0}^{n+1} a_n^{-1} a_i x^i$, and assume $r \in R$ is a root of $f(x)$. Then $f(r) = 0$, and so

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = 0.$$

Multiplying both sides by a_n^{-1} gives us

$$r^n + a_n^{-1} a_{n-1} r^{n-1} + \cdots + a_n^{-1} a_1 r + a_n^{-1} a_0 = a_n^{-1} 0 = 0.$$

But the left hand side of the last equation is just $g(r)$. Thus, $g(r) = 0$.

Now assume $r \in R$ is a root of $g(x)$. Then $g(r) = 0$. So

$$r^n + a_n^{-1} a_{n-1} r^{n-1} + \cdots + a_n^{-1} a_1 r + a_n^{-1} a_0 = 0.$$

Multiplying both sides by a_n yields

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 = a_n 0 = 0.$$

But the left hand side of the last equation is just $f(r)$. Thus, $f(r) = 0$. What we have shown is that r is a root of $f(x)$ if and only if r is a root of $g(x)$.

One conclusion we can draw from this is that when looking for roots of polynomials over a field, it is enough to consider roots of monic polynomials. So when finding roots of cubics, it suffices to consider only cubics of the form $x^3 + ax^2 + bx + c$. Let $a, b, c \in \mathbb{C}$, and consider the cubic equation

$$x^3 + ax^2 + bx + c = 0. \tag{E.1}$$

Our first step in solving this cubic equation is to reduce the cubic polynomial $x^3 + ax^2 + bx + c$ to what is often called a *depressed* cubic—that is, a cubic of the form $x^3 + px + q$. One way to do this is to make the change of variable $x = z - \frac{a}{3}$.

Activity E.1.

(a) Evaluate the cubic polynomial $x^3 + 6x^2 + x + 3$ at $x = z - \frac{a}{3}$ and show that the result is the depressed cubic $z^3 - 11z + 17$.

(b) Evaluate the general cubic $x^3 + ax^2 + bx + c$ at $x = z - \frac{a}{3}$ and show that

$$x^3 + ax^2 + bx + c = z^3 + \left(\frac{3b - a^2}{3}\right) z + \left(\frac{2a^3 - 9ab + 27c}{27}\right).$$

Activity E.1 shows that the substitution $x = z - \frac{a}{3}$ transforms our general cubic $x^3 + ax^2 + bx + c$ to the depressed cubic $z^3 + pz + q = 0$ with $p = \frac{3b-a^2}{3}$ and $q = \frac{2a^3-9ab+27c}{27}$. Now we just need to solve the depressed cubic.

Theorem E.2. *Let $p \in \mathbb{C}$ be nonzero, and let $q \in \mathbb{C}$. The roots of*

$$z^3 + pz + q = 0 \tag{E.2}$$

are given by $z = \sqrt[3]{A} - \frac{p}{3\sqrt[3]{A}}$, where $A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}$ and $\sqrt[3]{A}$ can be any one of the three cube roots of A in $\mathbb{C}$.

Proof. Our first step will be to reduce equation (E.2) to a quadratic. We do this by substituting $y - \frac{p}{3y}$ for z to obtain

$$\begin{aligned}
\left(y - \frac{p}{3y}\right)^3 + p\left(y - \frac{p}{3y}\right) + q &= 0 \\
y^3 - 3\left(\frac{p}{3y}\right)y^2 + 3\left(\frac{p}{3y}\right)^2 y - \left(\frac{p}{3y}\right)^3 + py - p\left(\frac{p}{3y}\right) + q &= 0 \\
y^3 - py + \left(\frac{p^2}{3y}\right) - \left(\frac{p^3}{27y^3}\right) + py - \left(\frac{p^2}{3y}\right) + q &= 0 \\
y^3 - \left(\frac{p^3}{27}\right)\left(\frac{1}{y^3}\right) + q &= 0 \\
\left(y^3\right)^2 + qy^3 - \left(\frac{p^3}{27}\right) &= 0. \qquad \text{(E.3)}
\end{aligned}$$

Setting $v = y^3$ transforms the last equation into the quadratic equation

$$v^2 + qv - \left(\frac{p^3}{27}\right) = 0. \tag{E.4}$$

We can solve equation (E.4) with the quadratic formula, which yields the following two roots:

$$A = \frac{-q + \sqrt{q^2 + 4\left(\frac{p^3}{27}\right)}}{2} \qquad \text{and} \qquad B = \frac{-q - \sqrt{q^2 + 4\left(\frac{p^3}{27}\right)}}{2}.$$

After some simplifying, we see that

$$A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} \qquad \text{and} \qquad B = -\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}.$$

So our solutions to equation (E.3) are the cube roots of A and B. At first glance it might appear that there are then 6 solutions to equation (E.4) and therefore 6 solutions to equation (E.2). However, if r is a cube root of A, then

$$\left(-\frac{p}{3r}\right)^3 = -\frac{p^3}{27r^3} = \frac{p^3}{27A}.$$

Since

$$AB = \frac{q^2}{4} - \left(\frac{q^2}{4} + \frac{p^3}{27}\right) = \frac{p^3}{27},$$

we see that

$$\left(-\frac{p}{3r}\right)^3 = -\frac{p^3}{27r^3} = \frac{p^3}{27A} = B,$$

and so $-\frac{p}{3r}$ is a cube root of B. So if we let r_1, r_2, and r_3 be the cube roots of A, then $s_1 = -\frac{p}{3r_1}$, $s_2 = -\frac{p}{3r_2}$, and $s_3 = -\frac{p}{3r_3}$ are the cube roots of B. Since $z = y - \frac{p}{3y}$ is equal to $r_i + s_i$ whenever $y = r_i$ or $y = s_i$, it follows that the solutions to equation (E.2) are $r_1 + s_1$, $r_2 + s_2$, and $r_3 + s_3$. ∎

To illustrate the cubic formula, we will find the solutions to the cubic equation $x^3 + x^2 + x + 1 = 0$. We begin by reducing this equation to the depressed cubic

$$z^3 + \frac{2}{3}z + \frac{20}{27} = 0$$

by substituting $x = z - \frac{1}{3}$, where $p = \frac{3(1)-(1)^2}{3} = \frac{2}{3}$ and $q = \frac{2(1)^3 - 9(1)(1) + 27(1)}{27} = \frac{20}{27}$. To solve the depressed cubic, we need to find the cube roots of

$$A = -\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}} = -\frac{10}{27} + \frac{2\sqrt{3}}{9}.$$

Let $r = \sqrt[3]{A}$, and let ω be the primitive cube root* of 1 given by $\omega = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. Then, using the same notation as in the proof of Theorem E.2, we obtain:

$$\begin{aligned}
r_1 &= r\omega & s_1 &= -\frac{p}{3r_1} = -\frac{2}{9r\omega} = -\frac{2}{9r}\omega^2 \\
r_2 &= r\omega^2 & s_2 &= -\frac{p}{3r_2} = -\frac{2}{9r}\omega \\
r_3 &= r & s_3 &= -\frac{p}{3r_3} = -\frac{2}{9r}.
\end{aligned}$$

So the roots of the depressed polynomial $z^3 + \frac{2}{3}z + \frac{20}{27}$ are

$$\begin{aligned}
z_1 &= r\omega - \frac{2}{9r}\omega^2 \\
z_2 &= r\omega^2 - \frac{2}{9r}\omega \\
z_3 &= r - \frac{2}{9r}.
\end{aligned}$$

We can simplify these roots a bit. First, note that $\left(-\frac{1}{3} + \frac{\sqrt{3}}{3}\right)^3 = -\frac{10}{27} + \frac{2\sqrt{3}}{9}$, so $r = -\frac{1}{3} + \frac{\sqrt{3}}{3}$.

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*If A is a real number and $\omega = \cos(\frac{2\pi}{n}) + i\sin(\frac{2\pi}{n})$, then ω is called a *primitive* n^{th} root of 1, and the n complex n^{th} roots of A are given by $\sqrt[n]{A}, \omega\sqrt[n]{A}, \omega^2\sqrt[n]{A}, \ldots, \omega^{n-1}\sqrt[n]{A}$.

Then

$$\begin{aligned}
z_3 &= r - \frac{2}{9r} \\
&= -\frac{1}{3} + \frac{\sqrt{3}}{3} - \frac{2}{9\left(-\frac{1}{3} + \frac{\sqrt{3}}{3}\right)} \\
&= -\frac{1}{3} + \frac{\sqrt{3}}{3} - \frac{2}{9\left(-\frac{1}{3} + \frac{\sqrt{3}}{3}\right)} \left(\frac{-\frac{1}{3} - \frac{\sqrt{3}}{3}}{-\frac{1}{3} - \frac{\sqrt{3}}{3}}\right) \\
&= -\frac{1}{3} + \frac{\sqrt{3}}{3} - \frac{2\left(-\frac{1}{3} - \frac{\sqrt{3}}{3}\right)}{9\left(\frac{1}{9} - \frac{3}{9}\right)} \\
&= -\frac{1}{3} + \frac{\sqrt{3}}{3} + \left(-\frac{1}{3} - \frac{\sqrt{3}}{3}\right) \\
&= -\frac{2}{3}.
\end{aligned}$$

Similar simplification gives $z_1 = \frac{1}{3} + i$ and $z_2 = \frac{1}{3} - i$. The solutions to our original equation $x^3 + x^2 + x + 1 = 0$ have the form $x_i = z_i - \frac{a}{3} = z_i - \frac{1}{3}$. So the solutions to $x^3 + x^2 + x + 1 = 0$ are

$$x_1 = i, \quad x_2 = -i, \text{ and } \quad x_3 = -1.$$

Theorem E.2 shows us how to find the roots of cubic polynomials in $\mathbb{C}[x]$. There is a corresponding formula for finding roots of quartic (degree 4) polynomials as well, but we won't consider that formula here. When we use the quadratic and cubic formulas to solve equations, we are finding the solutions in a form that only depends on the sums, differences, products, or quotients of the coefficients of the polynomial along with roots (square, cube, etc.) of such combinations of the coefficients. When we do this, we say we are *solving an equation by radicals*. Some of the best mathematicians throughout history, including Euler and Lagrange, attempted to find solutions by radicals of general quintic (degree 5) polynomial equations over $\mathbb{C}$. It wasn't until 1826 that the first generally accepted proof of the insolvability of quintic polynomials was published by Abel. Galois later developed a theory of solvability of equations involving groups and fields, and he used this theory to show that polynomial equations of degree 5 or higher over $\mathbb{C}$ are not solvable by radicals.