Section 9

Sequences in Metric Spaces

Focus Questions

By the end of this section, you should be able to give precise and thorough answers to the questions listed below. You may want to keep these questions in mind to focus your thoughts as you complete the section.

- What is a sequence in a metric space?
- What does it mean for a sequence to have a limit in a metric space?
- How can we use sequences to determine the continuity of a function at a point?

Introduction

We were introduced to sequences in calculus, and we can extend the notion of the limit of a sequence to metric spaces. Sequences provide an alternate way to describe many ideas in metric spaces. For example, we will see that we can characterize continuity in terms of sequences, and we can use sequences to determine open and closed sets.

Recall from calculus that a sequence of real numbers is a list of numbers in a specified order. We write a sequence a_1, a_2, $\ldots$, a_n, $\ldots$ as $(a_n)_{n\in\mathbb{Z}^+}$ or just (a_n). If we think of each a_n as the output of a function, we can give a more formal definition of a sequence as a function $f : \mathbb{Z}^+ \to \mathbb{R}$, where $a_n = f(n)$ for each n.

A sequence (a_n) of real numbers converges to a number L if we can make all of the numbers in the sequence as close to L as we like by choosing n to be large enough. Once again, this is an informal description that we need to make more rigorous. As we saw with continuous functions, we can make more rigorous the idea of "closeness" by introducing a symbol for a number that can be arbitrarily small. So we can say that the numbers a_n can get as close to a number L as we want if we can make $|a_n - L| < \epsilon$ for any positive number ϵ. The idea of choosing n large enough is just finding a large enough fixed integer N so that $|a_n - L| < \epsilon$ whenever $n \geq N$. This leads to the definition.

Definition 9.1. A sequence (a_n) of real numbers has a **limit** L if, given any $\epsilon > 0$ there exists a positive integer N (depending only on ϵ) such that

$$|a_n - L| < \epsilon \text{ whenever } n \geq N.$$

When a sequence (a_n) has a limit L, we write

$$\lim_{n\to\infty} a_n = L,$$

or just $\lim a_n = L$ (since we assume the limit for a sequence occurs as n goes to infinity) and we say that the sequence (a_n) *converges* to L.

Example 9.2. We can draw a graph of a sequence (a_n) of real numbers as the set of points (n, a_n). In this way we can visualize a sequence and its limit. By definition, L is a limit of the sequence (a_n) if, given any $\epsilon > 0$, we can go far enough out in the sequence so that the numbers in the sequence all lie in the horizontal band between $y = L - \epsilon$ and $L + \epsilon$ as illustrated in Figure 9.1 for the sequence $\left(\frac{n}{1+n}\right)$. To verify that the limit of the sequence $\left(\frac{n}{1+n}\right)$ is 1, we start by letting ϵ be an

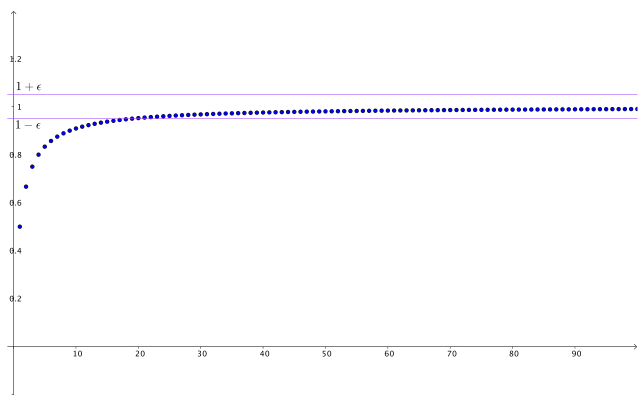

Figure 9.1: The limit of the sequence $\left(\frac{n}{1+n}\right)$.

arbitrary (small) positive number.

> **Scratch work.** Now we need to find N so that $n \geq N$ implies $\left|\frac{n}{1+n} - 1\right| < \epsilon$. Just as with our continuity example earlier, this work is not part of the proof, but shows how we go about finding the N we need. To make $\left|\frac{n}{1+n} - 1\right| < \epsilon$ we need

$$\begin{aligned} \left|\frac{n}{1+n} - 1\right| &< \epsilon \\ \left|\frac{n}{1+n} - \frac{1+n}{1+n}\right| &< \epsilon \\ \left|\frac{-1}{1+n}\right| &< \epsilon \\ 1 + n &> \frac{1}{\epsilon} \\ n &> \frac{1}{\epsilon} - 1. \end{aligned}$$

Now we use this scratch work to design our proof.

Let $N > \frac{1}{\epsilon} - 1$ (so that N depends on ϵ). Then for $n \geq N$ we have

$$\begin{aligned} n \geq N &> \frac{1}{\epsilon} - 1 \\ 1 + n &> \frac{1}{\epsilon} \\ |-1|\left|\frac{1}{1+n}\right| &< \epsilon \\ \left|\frac{-1}{1+n}\right| &< \epsilon \\ \left|\frac{n}{1+n} - 1\right| &< \epsilon \\ \left|\frac{n}{1+n} - \frac{1+n}{1+n}\right| &< \epsilon. \end{aligned}$$

So the sequence $\left(\frac{n}{1+n}\right)$ has a limit of 1.

Definition 9.1 only applies to sequences of real numbers. Ultimately, we want to phrase the definition in a way that allows us to define limits of sequences in metric spaces and topological spaces. So we have to reformulate the definition in such a way that it does not depend on distances.

Recall that $|x - y|$ defined a metric d_E on $\mathbb{R}$, that is

$$d_E(x, y) = |x - y|.$$

So we can rephrase the definition of a limit of a sequence of real numbers as follows.

Definition 9.3 (Alternate Definition)**.** A sequence (a_n) of real numbers has a **limit** L if, given any $\epsilon > 0$ there exists a positive integer N (depending only on ϵ) such that

$$d_E(a_n, L) < \epsilon \text{ whenever } n \geq N.$$

Once we have described a limit of a sequence in terms of a metric, then we can extend the idea into any metric space.

Definition 9.4. A **sequence** in a metric space (X, d) is a function $f : \mathbb{Z}^+ \to X$.

If f is a sequence in X, we write the sequence defined by f as $(f(n))$, where $n \in \mathbb{Z}^+$. We also use the notation (a_n), when $a_n = f(n)$. As long as X has a metric defined on it, we can then describe the limit of a sequence.

Definition 9.5. Let (X, d) be a metric space. A sequence (a_n) in X has a **limit** $L \in X$ if, given any $\epsilon > 0$ there exists a positive integer N (depending only on ϵ) such that

$$d(a_n, L) < \epsilon \text{ whenever } n \geq N.$$

In other words, a sequence (a_n) in a metric space (X, d) has a limit $L \in X$ if $\lim d(a_n, L) = 0$ – or that the sequence $d(a_n, L)$ of real numbers has a limit of 0. Just as with sequences of real numbers, when a sequence (a_n) has a limit L, we say that the sequence (a_n) *converges* to L, or that L is a limit of the sequence (a_n).

Preview Activity 9.1.

(1) Explain why the sequence $\left(\frac{1}{n}\right)$ converges to 0 in $\mathbb{R}$ using the Euclidean metric d_E, where

$$d_E(a, y) = |x - y|.$$

(2) Consider the sequence $(a_n) = \left(\left(\frac{1}{n}, \frac{1}{n+1}\right)\right)$ in $(\mathbb{R}^2, d_T)$, where d_T is the taxicab metric

$$d_T((x_1, x_2), (y_1, y_1)) = |x_1 - y_1| + |x_2 - y_2|.$$

Does the sequence (a_n) converge? If so, find its limit and prove that your candidate is the limit. If not, explain why.

(3) Let $(b_n) = \left((2n, n^2)\right)$ in the metric space $(\mathbb{R}^2, d)$, where d is the discrete metric defined by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Does the sequence (b_n) converge? If so, find its limit and prove that your candidate is the limit. If not, explain why.

Sequences and Continuity in Metric Spaces

We have seen that there are different ways to characterize. For example, there is the ϵ-δ definition and a characterization in terms of neighborhoods. In this section we investigate sequences and limits of sequences in metric spaces, and then provide a characterization of continuous functions in terms of sequences.

Activity 9.1. A reasonable question to ask is if a limit of a sequence is unique. In this activity we will show that the answer to this question is yes. Let (X, d) be a metric space and let (a_n) be a sequence in X. Assume the sequence (a_n) has a limit in X. To show that a limit of the sequence (a_n) is unique, we need to show that if $\lim a_n = a$ and $\lim a_n = a'$ for some $a, a' \in X$, then $a = a'$.

Suppose $\lim a_n = a$ and $\lim a_n = a'$ for some $a, a' \in X$. Without much to go on it might appear that proving $a = a'$ is a difficult task. However, if $d(a, a') < \epsilon$ for any $\epsilon > 0$, then it will have to be the case that $a = a'$. So let $\epsilon > 0$.

(a) Why must there exist a positive integer N so that $d(a_n, a) < \frac{\epsilon}{2}$ for all $n \geq N$?

(b) Why must there exist a positive integer N' so that $d(a_n, a') < \frac{\epsilon}{2}$ for all $n \geq N'$?

(c) Now let $m = \max\{N, N'\}$. What can we say about $d(a_m, a)$ and $d(a_m, a')$? Why?

(d) Use the triangle inequality to conclude that $d(a, a') < \epsilon$. Explain why this shows that $a = a'$.

Now we will examine how continuity can be described in terms of sequences. The basic idea is this. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point a. This means that f has a limit (as a continuous function) at a. So if we were to take any sequence (a_n) that converges to a, then the continuity of f implies that $f(a) = f(\lim a_n) = \lim f(a_n)$. That this is both a necessary condition and a sufficient condition for continuity is given in the next theorem.

Theorem 9.6. *Let (X, d_X) and (Y, d_Y) be metric spaces, and let $a \in X$. A function $f : X \to Y$ is continuous at a if and only if $\lim f(a_n) = f(a)$ for any sequence (a_n) in X that converges to a.*

Proof. Let (X, d_X) and (Y, d_Y) be metric spaces, let $a \in X$, and let $f : X \to Y$ be a function. Assume that f is continuous at a. We will show that $\lim f(a_n) = f(a)$ for any sequence (a_n) in X that converges to a. Let (a_n) be a sequence in X that converges to a (we know such a sequence exists, namely the sequence (a)). To verify that $\lim f(a_n) = a$, let $\epsilon > 0$. The fact that f is continuous at a means that there is a $\delta > 0$ so that $d_Y(f(x), f(a)) < \epsilon$ whenever $d_X(x, a) < \delta$. Since (a_n) converges to a, we know that there exists a positive integer N such that $d_X(a_n, a) < \delta$ whenever $n \geq N$. This implies that

$$d_Y(f(a_n), f(a)) < \epsilon \text{ whenever } n \geq N.$$

We conclude that if f is continuous at a, then $\lim f(a_n) = f(a)$ for any sequence (a_n) in X that converges to a.

The proof of the reverse implication is contained in the next activity. ■

Activity 9.2. Let (X, d_X) and (Y, d_Y) be metric spaces, let $a \in X$, and let $f : X \to Y$ be a function. We prove the remaining implication of Theorem 9.6, that f is continuous at a if $\lim f(a_n) = f(a)$ for any sequence (a_n) in X that converges to a, in this activity.

(a) To have an additional assumption with which to work, let us proceed by contradiction and assume that f is not continuous at a. Why can we then say that there is an $\epsilon > 0$ so that there is no $\delta > 0$ with the property that $d_X(x, a) < \delta$ implies $d_Y(f(x), f(a)) < \epsilon$?

(b) To create a contradiction, we will construct a sequence (a_n) that converges to a while $(f(a_n))$ does not converge to $f(a)$.

i. Explain why we can find a positive integer K such that $\frac{1}{K} < \epsilon$.

ii. If $k > K$, explain why there is an element $a_k \in B\left(a, \frac{1}{k}\right)$ so that $d_Y(f(a_k), f(a)) \geq \epsilon$.

iii. For $k \leq K$, let a_k be any element in $B\left(a, \frac{1}{k}\right)$. Explain why a is a limit of (a_n).

iv. Explain why $f(a)$ is not a limit of the sequence $(f(a_n))$. What conclusion can we draw, and why?

One way that Theorem 9.6 is often used is illustrated in the next activity.

Activity 9.3. Let f be the function from $\mathbb{R}$ to $\mathbb{R}$, both with the Euclidean metric, defined by

$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

We consider the f continuity of f at 0 in this activity.

(a) Draw a graph of f on some small interval centered at 0. Based on the graph, do you think f has a limit at 0? Explain. (There is no right answer here, just your intuition based on the graph.)

(b) At which inputs is $f(x) = 1$?

(c) Use the result of (b) to find a sequence (a_n) that converges to 0 for which $f(a_n) = 1$ for every n.

(d) What does the result of (c) tell us about the continuity of f at 0?

While it can sometimes be difficult to prove a fact about all sequences that converge to a point, Activity 9.3 shows that we can use Theorem 9.6 to prove that a function f is not continuous at an input a be finding just one sequence (a_n) that converges to a for which $\lim f(a_n) \neq f(a)$. We conclude this section with one final note.

IMPORTANT NOTE: Theorem 9.6 tells us that if $f : X \to Y$ is a continuous function, then f commutes with limits. That is, if (a_n) is a sequence in X that converges to $a \in X$, then

$$f(a) = f(\lim a_n) = \lim f(a_n).$$

Summary

Important ideas that we discussed in this section include the following.

- A sequence in a metric space X is a function $f : \mathbb{Z}^+ \to X$.
- A sequence (a_n) in a metric space (X, d) has a limit L in X if, given any $\epsilon > 0$ there exists a positive integer N such that $d(a_n, L) < \epsilon$ whenever $n \geq N$.
- Let f be a function from a metric space (X, d_X) to m metric space (Y, d_Y). Then f is continuous at $a \in X$ if and only if $\lim f(a_n) = f(a)$ for any sequence (a_n) in X that converges to a.

Exercises

(1) Determine, with proof, the convergence or divergence of each of the following sequences in the indicated metric spaces.

(a) $a_n = 1 + \frac{1}{n}$ in $(\mathbb{R}, d_E)$

(b) $a_n = (2, n)$ in $(\mathbb{R}^2, d_M)$

(c) a_n is the function defined by

$$a_n(x) = \frac{1}{n}x$$

where X is the set of real valued functions on the interval $[0, 1]$ and the metric d is defined by

$$d(f, g) = \sup\{|f(x) - g(x)| \mid x \in [0, 1]\}.$$

(See Exercise 4 on page 53.)

(2) Let A be a subset of $\mathbb{R}$.

(a) Show that if A is bounded above, then there is a sequence (a_n) in A such that $\lim a_n = \sup(A)$.

(b) Show that if A is bounded below, then there is a sequence (a_n) in A such that $\lim a_n = \inf(A)$.

(c) Are the limits from (a) or (b) necessarily in A? Explain.

(3) Let (X, d) be a metric space, let $x \in X$, and let A be a nonempty subset of X. Recall that the distance from x to A is

$$d(x, A) = \inf\{d(x, a) \mid a \in A.$$

In this exercise we see how we can view the distance between a point and a set in terms of sequences. Let $m = d(x, A)$. We will show that there must be a sequence (a_n) in A so that $d(x, A) = \lim d(x, a_n)$.

(a) For each $n \in \mathbb{Z}^+$, let $B_n = B\left(x; m + \frac{1}{n}\right)$. Why must $B_n \cap A \neq \emptyset$ for each $n \in \mathbb{Z}^+$?

(b) Let $a_n \in B_n \cap A$ for each n. What property does this sequence have? Explain how we have just proved the following theorem.

Theorem 9.7. *Let (X, d) be a metric space, let $x \in X$, and let A be a nonempty subset of X. Then there exists a sequence (a_n) in A such that*

$$\lim d(x, a_n) = d(x, A).$$

(4)

(a) Let (Y, d') be a subspace of (X, d). Let $a_1, a_2, \ldots$ be a sequence of points in Y and let $a \in Y$. Prove that if $\lim_n a_n = a$ in (Y, d'), then $\lim_n a_n = a$ in (X, d).

(b) Show that the converse of part (a) is false by considering the subspace $(\mathbb{Q}, d_{\mathbb{Q}})$ (the rational numbers) of (R, d). Let $a_1, a_2, \ldots$ be a sequence of rational numbers such that $\lim_n a_n = \sqrt{2}$. Prove that , given $\epsilon > 0$, there is a positive integer N such that for $n, m > N$, $|a_n - a_m| < \epsilon$. Does the sequence $a_1, a_2, \ldots$ converge when considered to be a sequence of points in $\mathbb{Q}$?

(5) In this exercise we prove some standard results about limits of sequences from calculus. Let (a_n) and (b_n) be convergent sequences in a metric space $(\mathbb{R}, d_E)$.

(a) Show that $\lim ka_n = k \lim a_n$ for any real number k.

(b) Show that $\lim(a_n + b_n) = \lim a_n + \lim b_n$.

(c) Show that the sequence (a_n) is bounded. That is, show that there is a positive real number M such that $|a_n| \leq M$ for all $n \in \mathbb{Z}^+$.

(d) Show that $\lim a_n b_n = \lim a_n \ \lim b_n$.

(e) If $b_n \neq 0$ for every n and $\lim b_n \neq 0$, show that $\lim \frac{a_n}{b_n} = \frac{\lim a_n}{\lim b_n}$.

(6) Let f and g be continuous functions from $\mathbb{R}$ to $\mathbb{R}$, both with the standard Euclidean metric. Define the function fg from $\mathbb{R}$ to $\mathbb{R}$ by

$$(fg)(x) = f(x)g(x) \text{ for every } x \in \mathbb{R}.$$

(a) Prove that fg is a continuous function.

(b) Assume that $g(x) \neq 0$ for every $x \in \mathbb{R}$. Define the function $\frac{f}{g}$ from $\mathbb{R}$ to $\mathbb{R}$ by $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$ for every $x \in \mathbb{R}$. Use the definition of continuity to prove that $\frac{f}{g}$ is a continuous function.

(7) Let $(c_n) = (a_n, b_n)$ be a sequence in $(\mathbb{R}^2, d_E)$. Show that the sequence (c_n) converges to a point (a, b) if and only if (a_n) converges to a and (b_n) converges to b in $(\mathbb{R}, d_E)$.

(8) Define $f : (\mathbb{R}, d_E) \to (\mathbb{R}, d_E)$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational} \\ x & \text{if } x \text{ is rational.} \end{cases}$$

(a) Show that f is continuous at exactly one point. Assume that both copies of $\mathbb{R}$ are given the Euclidean topology.

(b) Modify the function f to construct a new function $g : \mathbb{R} \to \mathbb{R}$ such that g is continuous at exactly the numbers 0 and 1. Prove your result. Can you see how to extend this to construct a function $h : \mathbb{R} \to \mathbb{R}$ that is continuous at any given finite number of points?

(9) Let X be the set of real valued functions on the interval $[0, 1]$ and let d be the metric on X defined by

$$d(f, g) = \sup\{|f(x) - g(x)| \mid x \in [0, 1]\}.$$

(See Exercise 4 on page 53.)

There is a difference between the point-wise convergence of a sequence of functions and convergence in the metric space (X, d) that we explore in this exercise. For each $n \in \mathbb{Z}^+$, define $f_n : [0, 1] \to \mathbb{R}$ by $f_n(x) = x^n$.

(a) Let $0 \leq a < 1$. Show that the sequence (a_n) where $a_n = a^n$ converges to 0 in $(\mathbb{R}, d_E)$.

(b) Since the sequence (1) converges to 1, if we look at the behavior at each point, we might think that the sequence (f_n) converges to the function f defined by

$$f(x) = \begin{cases} 0 & \text{if } x \neq 1 \\ 1 & \text{if } x = 1. \end{cases} \tag{9.1}$$

Determine if the sequence (f_n) converges to (f) in the metric space (X, d).

(c) Suppose now we consider the sequence (f_n) as a sequence of functions in $C[0,1]$, the space of continuous functions from $\mathbb{R}$ to $\mathbb{R}$, using the metric

$$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx.$$

(Refer to Activity 3.1.) The function in (9.1) is not a continuous function, so can't be a limit of the sequence (f_n) in $C[01]$. Determine if the sequence (f_n) has a limit in $C[0,1]$. If so, what is the limit? If not, verify that the sequence has no limit.

(10) For each of the following, answer true if the statement is always true. If the statement is only sometimes true or never true, answer false and provide a concrete example to illustrate that the statement is false. If a statement is true, explain why.

(a) If (a_n) is a sequence in $(\mathbb{R}, d_E)$ with $a_{n+1} < a_n$ for each $n \in \mathbb{Z}^+$ and the set $\{a_n\}$ is bounded below, then $\inf\{a_n \mid n \in \mathbb{Z}^+\}$ is the limit of the sequence (a_n).

(b) Let X be a metric space and A a nonempty subset of X. If $a \in X$ and if $B(a,r)$ in X contains a point of A for every $r > 0$, then there is a sequence in A that converges to a.

(c) Let R be a nonempty subset of $\mathbb{R}$ that is bounded above and below. If S is a nonempty subset of $\mathbb{R}$ and $x \leq y$ for all $x \in S$ and for all $y \in R$, then $\sup(S) \leq \inf(R)$.

(d) The sequence $\left(\frac{1}{n}\right)$ converges to 0 in the metric space Q of all rational numbers in reduced form with metric d defined by

$$d\left(\frac{a}{b}, \frac{r}{s}\right) = \max\{|a-r|, |b-s|\}.$$

(See Exercise 3 on page 40.)

(e) The only convergent sequences in a metric space (X,d) with discrete metric d are the sequences that are eventually constant. (A sequence (a_n) in a metric space X is eventually constant if there is an element $a \in X$ and an $N \in \mathbb{Z}^+$ such that $a_n = a$ for all $n \geq N$.)