

Chapter 5

Finding Antiderivatives and Evaluating Integrals

5.1 Constructing Accurate Graphs of Antiderivatives

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- Given the graph of a function's derivative, how can we construct a completely accurate graph of the original function?
- How many antiderivatives does a given function have? What do those antiderivatives all have in common?
- Given a function f , how does the rule $A(x) = \int_0^x f(t) dt$ define a new function A ?

Introduction

A recurring theme in our discussion of differential calculus has been the question “Given information about the derivative of an unknown function f , how much information can we obtain about f itself?” For instance, in Activity 1.22, we explored the situation where the graph of $y = f'(x)$ was known (along with the value of f at a single point) and endeavored to sketch a possible graph of f near the known point. In Example 3.1 – and indeed throughout Section 3.1 – we investigated how the first derivative test enables us to use information regarding f' to determine where the original function f is increasing and decreasing, as well as where f has relative extreme values. Further, if we know a formula or graph of f' , by computing f'' we can find where the original function f is concave

up and concave down. Thus, the combination of knowing f' and f'' enables us to fully understand the shape of the graph of f .

We returned to this question in even more detail in Section 4.1; there, we considered the situation where we knew the instantaneous velocity of a moving object and worked from that information to determine as much information as possible about the object's position function. We found key connections between the net-signed area under the velocity function and the corresponding change in position of the function; in Section 4.4, the Total Change Theorem further illuminated these connections between f' and f in a more general setting, such as the one found in Figure 4.34, showing that the total change in the value of f over an interval $[a, b]$ is determined by the exact net-signed area bounded by f' and the x -axis on the same interval.

In what follows, we explore these issues still further, with a particular emphasis on the situation where we possess an accurate graph of the derivative function along with a single value of the function f . From that information, we desire to completely determine an accurate graph of f that not only represents correctly where f is increasing, decreasing, concave up, and concave down, but also allows us to find an accurate function value at any point of interest to us.

Preview Activity 5.1. Suppose that the following information is known about a function f : the graph of its derivative, $y = f'(x)$, is given in Figure 5.1. Further, assume that f' is piecewise linear (as pictured) and that for $x \leq 0$ and $x \geq 6$, $f'(x) = 0$. Finally, it is given that $f(0) = 1$.

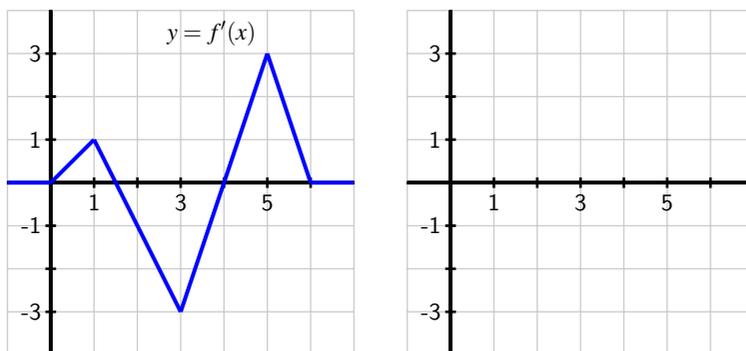


Figure 5.1: At left, the graph of $y = f'(x)$; at right, axes for plotting $y = f(x)$.

- On what interval(s) is f an increasing function? On what intervals is f decreasing?
- On what interval(s) is f concave up? concave down?
- At what point(s) does f have a relative minimum? a relative maximum?

(d) Recall that the Total Change Theorem tells us that

$$f(1) - f(0) = \int_0^1 f'(x) dx.$$

What is the exact value of $f(1)$?

(e) Use the given information and similar reasoning to that in (d) to determine the exact value of $f(2)$, $f(3)$, $f(4)$, $f(5)$, and $f(6)$.

(f) Based on your responses to all of the preceding questions, sketch a complete and accurate graph of $y = f(x)$ on the axes provided, being sure to indicate the behavior of f for $x < 0$ and $x > 6$.

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Constructing the graph of an antiderivative

Preview Activity 5.1 demonstrates that when we can find the exact area under a given graph on any given interval, it is possible to construct an accurate graph of the given function's antiderivative: that is, we can find a representation of a function whose derivative is the given one. While we have considered this question at different points throughout our study, it is important to note here that we now can determine not only the overall shape of the antiderivative, but also the actual *height* of the antiderivative at any point of interest.

Indeed, this is one key consequence of the Fundamental Theorem of Calculus: if we know a function f and wish to know information about its antiderivative, F , provided that we have some starting point a for which we know the value of $F(a)$, we can determine the value of $F(b)$ via the definite integral. In particular, since $F(b) - F(a) = \int_a^b f(x) dx$, it follows that

$$F(b) = F(a) + \int_a^b f(x) dx. \quad (5.1)$$

Moreover, in the discussion surrounding Figure 4.34, we made the observation that differences in heights of a function correspond to net-signed areas bounded by its derivative. Rephrasing this in terms of a given function f and its antiderivative F , we observe that on an interval $[a, b]$,

differences in heights on the antiderivative (such as $F(b) - F(a)$) correspond to the net-signed area bounded by the original function on the interval $[a, b]$ ($\int_a^b f(x) dx$).

For example, say that $f(x) = x^2$ and that we are interested in an antiderivative of f that satisfies $F(1) = 2$. Thinking of $a = 1$ and $b = 2$ in Equation (5.1), it follows from the

Fundamental Theorem of Calculus that

$$\begin{aligned}
 F(2) &= F(1) + \int_1^2 x^2 dx \\
 &= 2 + \left. \frac{1}{3}x^3 \right|_1^2 \\
 &= 2 + \left(\frac{8}{3} - \frac{1}{3} \right) \\
 &= \frac{13}{3}.
 \end{aligned}$$

In this way, we see that if we are given a function f for which we can find the exact net-signed area bounded by f on a given interval, along with one value of a corresponding antiderivative F , we can find any other value of F that we seek, and in this way construct a completely accurate graph of F . We have two main options for finding the exact net-signed area: using the Fundamental Theorem of Calculus (which requires us to find an algebraic formula for an antiderivative of the given function f), or, in the case where f has nice geometric properties, finding net-signed areas through the use of known area formulas.

Activity 5.1.

Suppose that the function $y = f(x)$ is given by the graph shown in Figure 5.2, and that the pieces of f are either portions of lines or portions of circles. In addition, let F be an antiderivative of f and say that $F(0) = -1$. Finally, assume that for $x \leq 0$ and $x \geq 7$, $f(x) = 0$.

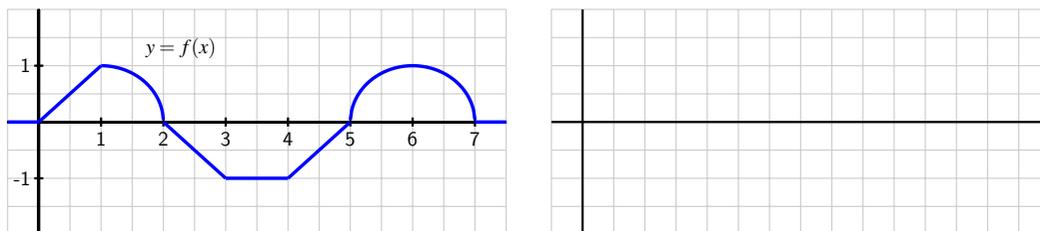


Figure 5.2: At left, the graph of $y = f(x)$.

- On what interval(s) is F an increasing function? On what intervals is F decreasing?
- On what interval(s) is F concave up? concave down? neither?
- At what point(s) does F have a relative minimum? a relative maximum?
- Use the given information to determine the exact value of $F(x)$ for $x = 1, 2, \dots, 7$. In addition, what are the values of $F(-1)$ and $F(8)$?

- (e) Based on your responses to all of the preceding questions, sketch a complete and accurate graph of $y = F(x)$ on the axes provided, being sure to indicate the behavior of F for $x < 0$ and $x > 7$. Clearly indicate the scale on the vertical and horizontal axes of your graph.
- (f) What happens if we change one key piece of information: in particular, say that G is an antiderivative of f and $G(0) = 0$. How (if at all) would your answers to the preceding questions change? Sketch a graph of G on the same axes as the graph of F you constructed in (e).

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Multiple antiderivatives of a single function

In the final question of Activity 5.1, we encountered a very important idea: a given function f has more than one antiderivative. In addition, any antiderivative of f is determined uniquely by identifying the value of the desired antiderivative at a single point. For example, suppose that f is the function given at left in Figure 5.3, and we say that F is an

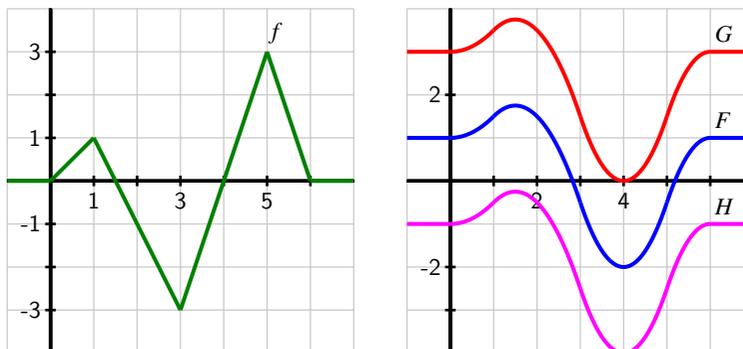


Figure 5.3: At left, the graph of $y = f(x)$. At right, three different antiderivatives of f .

antiderivative of f that satisfies $F(0) = 1$.

Then, using Equation 5.1, we can compute $F(1) = 1.5$, $F(2) = 1.5$, $F(3) = -0.5$, $F(4) = -2$, $F(5) = -0.5$, and $F(6) = 1$, plus we can use the fact that $F' = f$ to ascertain where F is increasing and decreasing, concave up and concave down, and has relative extremes and inflection points. Through work similar to what we encountered in Preview Activity 5.1 and Activity 5.1, we ultimately find that the graph of F is the one given in blue in Figure 5.3.

If we instead chose to consider a function G that is an antiderivative of f but has the property that $G(0) = 3$, then G will have the exact same shape as F (since both share the

derivative f), but G will be shifted vertically away from the graph of F , as pictured in red in Figure 5.3. Note that $G(1) - G(0) = \int_0^1 f(x) dx = 0.5$, just as $F(1) - F(0) = 0.5$, but since $G(0) = 3$, $G(1) = G(0) + 0.5 = 3.5$, whereas $F(1) = F(0) + 0.5 = 1.5$, since $F(0) = 1$. In the same way, if we assigned a different initial value to the antiderivative, say $H(0) = -1$, we would get still another antiderivative, as shown in magenta in Figure 5.3.

This example demonstrates an important fact that holds more generally:

If G and H are both antiderivatives of a function f , then the function $G - H$ must be constant.

To see why this result holds, observe that if G and H are both antiderivatives of f , then $G' = f$ and $H' = f$. Hence, $\frac{d}{dx}[G(x) - H(x)] = G'(x) - H'(x) = f(x) - f(x) = 0$. Since the only way a function can have derivative zero is by being a constant function, it follows that the function $G - H$ must be constant.

Further, we now see that if a function has a single antiderivative, it must have infinitely many: we can add any constant of our choice to the antiderivative and get another antiderivative. For this reason, we sometimes refer to the *general antiderivative* of a function f . For example, if $f(x) = x^2$, its general antiderivative is $F(x) = \frac{1}{3}x^3 + C$, where we include the “ $+C$ ” to indicate that F includes *all* of the possible antiderivatives of f . To identify a particular antiderivative of f , we must be provided a single value of the antiderivative F (this value is often called an *initial condition*). In the present example, suppose that condition is $F(2) = 3$; substituting the value of 2 for x in $F(x) = \frac{1}{3}x^3 + C$, we find that

$$3 = \frac{1}{3}(2)^3 + C,$$

and thus $C = 3 - \frac{8}{3} = \frac{1}{3}$. Therefore, the particular antiderivative in this case is $F(x) = \frac{1}{3}x^3 + \frac{1}{3}$.

Activity 5.2.

For each of the following functions, sketch an accurate graph of the antiderivative that satisfies the given initial condition. In addition, sketch the graph of two additional antiderivatives of the given function, and state the corresponding initial conditions that each of them satisfy. If possible, find an algebraic formula for the antiderivative that satisfies the initial condition.

- (a) original function: $g(x) = |x| - 1$;
 initial condition: $G(-1) = 0$;
 interval for sketch: $[-2, 2]$
- (b) original function: $h(x) = \sin(x)$;
 initial condition: $H(0) = 1$;
 interval for sketch: $[0, 4\pi]$

$$(c) \text{ original function: } p(x) = \begin{cases} x^2, & \text{if } 0 < x \leq 1 \\ -(x-2)^2, & \text{if } 1 < x < 2; \\ 0 & \text{otherwise} \end{cases}$$

initial condition: $P(0) = 1$;
interval for sketch: $[-1, 3]$

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Functions defined by integrals

In Equation (5.1), we found an important rule that enables us to compute the value of the antiderivative F at a point b , provided that we know $F(a)$ and can evaluate the definite integral from a to b of f . Again, that rule is

$$F(b) = F(a) + \int_a^b f(x) dx.$$

In several examples, we have used this formula to compute several different values of $F(b)$ and then plotted the points $(b, F(b))$ to assist us in generating an accurate graph of F . That suggests that we may want to think of b , the upper limit of integration, as a variable itself. To that end, we introduce the idea of an *integral function*, a function whose formula involves a definite integral.

Given a continuous function f , we define the corresponding integral function A according to the rule

$$A(x) = \int_a^x f(t) dt. \quad (5.2)$$

Note particularly that because we are using the variable x as the independent variable in the function A , and x determines the other endpoint of the interval over which we integrate (starting from a), we need to use a variable other than x as the variable of integration. A standard choice is t , but any variable other than x is acceptable.

One way to think of the function A is as the “net-signed area from a up to x ” function, where we consider the region bounded by $y = f(t)$ on the relevant interval. For example, in Figure 5.4, we see a given function f pictured at left, and its corresponding area function (choosing $a = 0$), $A(x) = \int_0^x f(t) dt$ shown at right.

Note particularly that the function A measures the net-signed area from $t = 0$ to $t = x$ bounded by the curve $y = f(t)$; this value is then reported as the corresponding height on the graph of $y = A(x)$. It is even more natural to think of this relationship between f and A dynamically. At <http://gvsu.edu/s/cz>, we find a java applet¹ that brings the static picture in Figure 5.4 to life. There, the user can move the red point on the function f and see how the corresponding height changes at the light blue point on the graph of A .

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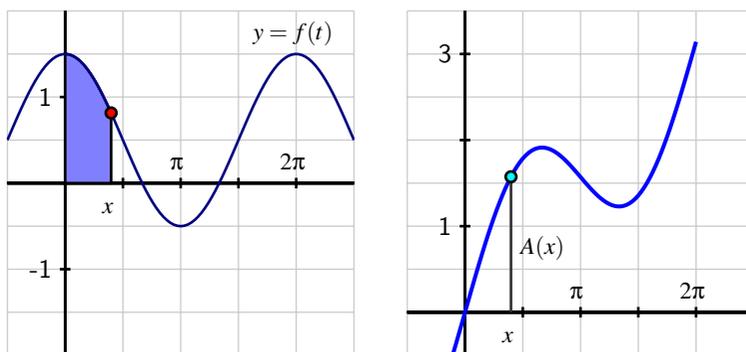


Figure 5.4: At left, the graph of the given function f . At right, the area function $A(x) = \int_0^x f(t) dt$.

The choice of a is somewhat arbitrary. In the activity that follows, we explore how the value of a affects the graph of the integral function, as well as some additional related issues.

Activity 5.3.

Suppose that g is given by the graph at left in Figure 5.5 and that A is the corresponding integral function defined by $A(x) = \int_1^x g(t) dt$.

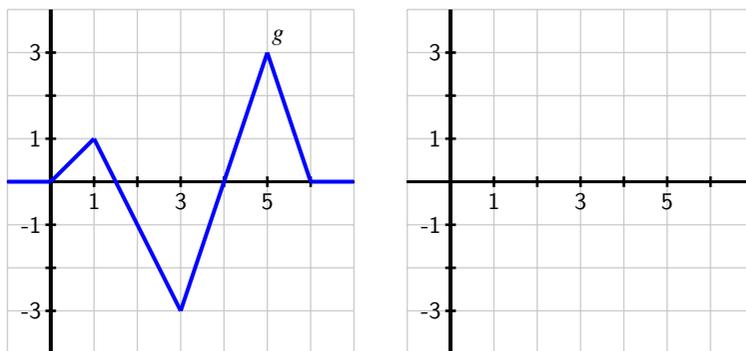


Figure 5.5: At left, the graph of $y = g(t)$; at right, axes for plotting $y = A(x)$, where A is defined by the formula $A(x) = \int_1^x g(t) dt$.

- On what interval(s) is A an increasing function? On what intervals is A decreasing? Why?
- On what interval(s) do you think A is concave up? concave down? Why?

- (c) At what point(s) does A have a relative minimum? a relative maximum?
- (d) Use the given information to determine the exact values of $A(0)$, $A(1)$, $A(2)$, $A(3)$, $A(4)$, $A(5)$, and $A(6)$.
- (e) Based on your responses to all of the preceding questions, sketch a complete and accurate graph of $y = A(x)$ on the axes provided, being sure to indicate the behavior of A for $x < 0$ and $x > 6$.
- (f) How does the graph of B compare to A if B is instead defined by $B(x) = \int_0^x g(t) dt$?

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Summary

In this section, we encountered the following important ideas:

- Given the graph of a function f , we can construct the graph of its antiderivative F provided that (a) we know a starting value of F , say $F(a)$, and (b) we can evaluate the integral $\int_a^b f(x) dx$ exactly for relevant choices of a and b . For instance, if we wish to know $F(3)$, we can compute $F(3) = F(a) + \int_a^3 f(x) dx$. When we combine this information about the function values of F together with our understanding of how the behavior of $F' = f$ affects the overall shape of F , we can develop a completely accurate graph of the antiderivative F .
- Because the derivative of a constant is zero, if F is an antiderivative of f , it follows that $G(x) = F(x) + C$ will also be an antiderivative of f . Moreover, any two antiderivatives of a function f differ precisely by a constant. Thus, any function with at least one antiderivative in fact has infinitely many, and the graphs of any two antiderivatives will differ only by a vertical translation.
- Given a function f , the rule $A(x) = \int_a^x f(t) dt$ defines a new function A that measures the net-signed area bounded by f on the interval $[a, x]$. We call the function A the integral function corresponding to f .

Exercises

1. A moving particle has its velocity given by the quadratic function v pictured in Figure 5.6. In addition, it is given that $A_1 = \frac{7}{6}$ and $A_2 = \frac{8}{3}$, as well as that for the corresponding position function s , $s(0) = 0.5$.
 - (a) Use the given information to determine $s(1)$, $s(3)$, $s(5)$, and $s(6)$.
 - (b) On what interval(s) is s increasing? On what interval(s) is s decreasing?
 - (c) On what interval(s) is s concave up? On what interval(s) is s concave down?

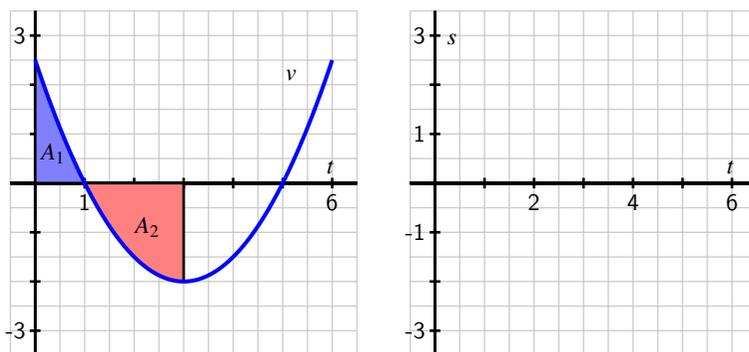


Figure 5.6: At left, the given graph of v . At right, axes for plotting s .

(d) Sketch an accurate, labeled graph of s on the axes at right in Figure 5.6.

(e) Note that $v(t) = -2 + \frac{1}{2}(t - 3)^2$. Find a formula for s .

2. A person exercising on a treadmill experiences different levels of resistance and thus burns calories at different rates, depending on the treadmill's setting. In a particular workout, the rate at which a person is burning calories is given by the piecewise constant function c pictured in Figure 5.7. Note that the units on c are "calories per minute."

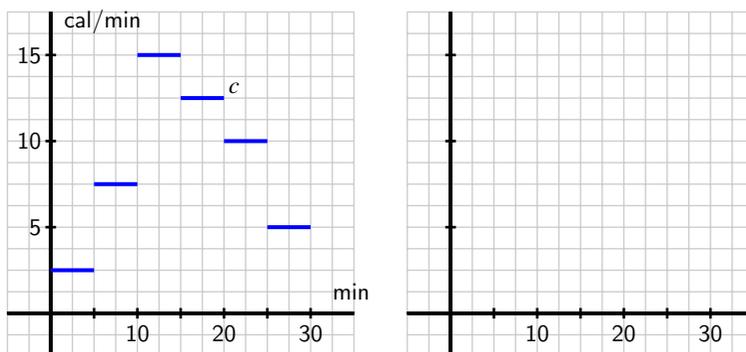


Figure 5.7: At left, the given graph of c . At right, axes for plotting C .

- (a) Let C be an antiderivative of c . What does the function C measure? What are its units?
- (b) Assume that $C(0) = 0$. Determine the exact value of $C(t)$ at the values $t = 5, 10, 15, 20, 25, 30$.

- (c) Sketch an accurate graph of C on the axes provided at right in Figure 5.7. Be certain to label the scale on the vertical axis.
- (d) Determine a formula for C that does not involve an integral and is valid for $5 \leq t \leq 10$.
3. Consider the piecewise linear function f given in Figure 5.8. Let the functions A , B , and C be defined by the rules $A(x) = \int_{-1}^x f(t) dt$, $B(x) = \int_0^x f(t) dt$, and $C(x) = \int_1^x f(t) dt$.

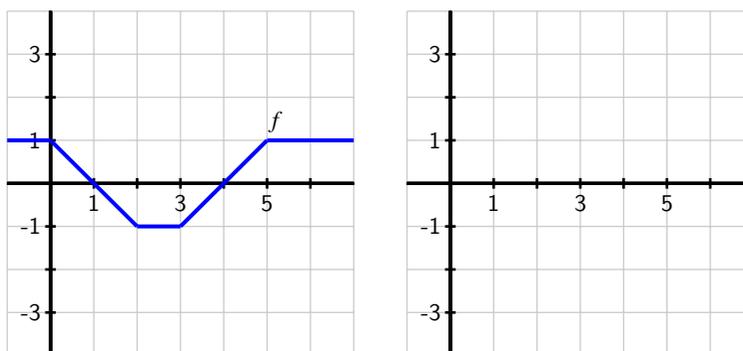


Figure 5.8: At left, the given graph of f . At right, axes for plotting A , B , and C .

- (a) For the values $x = -1, 0, 1, \dots, 6$, make a table that lists corresponding values of $A(x)$, $B(x)$, and $C(x)$.
- (b) On the axes provided in Figure 5.8, sketch the graphs of A , B , and C .
- (c) How are the graphs of A , B , and C related?
- (d) How would you best describe the relationship between the function A and the function f ?

5.2 The Second Fundamental Theorem of Calculus

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How does the integral function $A(x) = \int_1^x f(t) dt$ define an antiderivative of f ?
- What is the statement of the Second Fundamental Theorem of Calculus?
- How do the First and Second Fundamental Theorems of Calculus enable us to formally see how differentiation and integration are almost inverse processes?

Introduction

In Section 4.4, we learned the Fundamental Theorem of Calculus (FTC), which from here forward will be referred to as the *First* Fundamental Theorem of Calculus, as in this section we develop a corresponding result that follows it. In particular, recall that the First FTC tells us that if f is a continuous function on $[a, b]$ and F is any antiderivative of f (that is, $F' = f$), then

$$\int_a^b f(x) dx = F(b) - F(a).$$

We have typically used this result in two settings: (1) where f is a function whose graph we know and for which we can compute the exact area bounded by f on a certain interval $[a, b]$, we can compute the change in an antiderivative F over the interval; and (2) where f is a function for which it is easy to determine an algebraic formula for an antiderivative, we may evaluate the integral exactly and hence determine the net-signed area bounded by the function on the interval. For the former, see Preview Activity 5.1 or Activity 5.1. For the latter, we can easily evaluate exactly integrals such as

$$\int_1^4 x^2 dx,$$

since we know that the function $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$. Thus,

$$\begin{aligned} \int_1^4 x^2 dx &= \left. \frac{1}{3}x^3 \right|_1^4 \\ &= \frac{1}{3}(4)^3 - \frac{1}{3}(1)^3 \\ &= 21. \end{aligned}$$

Here we see that the First FTC can be viewed from at least two perspectives: first, as a tool to find the difference $F(b) - F(a)$ for an antiderivative F of the integrand f . In this situation, we need to be able to determine the value of the integral $\int_a^b f(x) dx$ exactly, perhaps through known geometric formulas for area. It is possible that we may not have a formula for F itself. From a second perspective, the First FTC provides a way to find the exact value of a definite integral, and hence a certain net-signed area exactly, by finding an antiderivative of the integrand and evaluating its total change over the interval. In this latter case, we need to know a formula for the antiderivative F , as this enables us to compute net-signed areas exactly through definite integrals, as demonstrated in Figure 5.9.

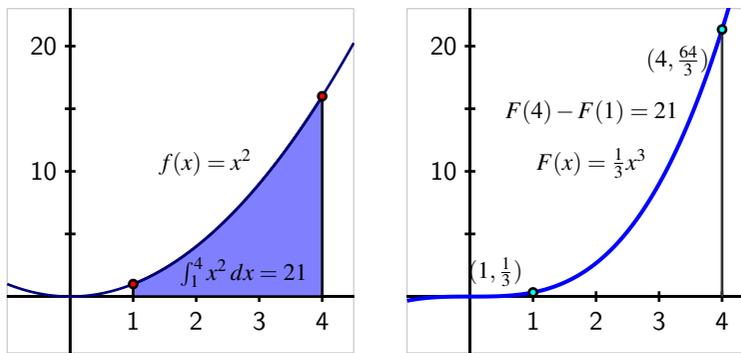


Figure 5.9: At left, the graph of $f(x) = x^2$ on the interval $[1, 4]$ and the area it bounds. At right, the antiderivative function $F(x) = \frac{1}{3}x^3$, whose total change on $[1, 4]$ is the value of the definite integral at left.

We recall further that the value of a definite integral may have additional meaning depending on context: change in position when the integrand is a velocity function, total pollutant leaked from a tank when the integrand is the rate at which pollution is leaking, or other total changes that correspond to a given rate function that is the integrand. In addition, the value of the definite integral is always connected to the average value of a continuous function on a given interval: $f_{\text{AVG}[a,b]} = \frac{1}{b-a} \int_a^b f(x) dx$.

Next, remember that in the last part of Section 5.1, we studied integral functions of the form $A(x) = \int_c^x f(t) dt$. Figure 5.4 is a particularly important image to keep in mind as we work with integral functions, and the corresponding java applet at <http://gvsu.edu/s/cz> is likewise foundational to our understanding of the function A . In what follows, we use the First FTC to gain additional understanding of the function $A(x) = \int_c^x f(t) dt$, where the integrand f is given (either through a graph or a formula), and c is a constant. In particular, we investigate further the special nature of the relationship between the functions A and f .

Preview Activity 5.2. Consider the function A defined by the rule

$$A(x) = \int_1^x f(t) dt,$$

where $f(t) = 4 - 2t$.

- Compute $A(1)$ and $A(2)$ exactly.
- Use the First Fundamental Theorem of Calculus to find an equivalent formula for $A(x)$ that does not involve integrals. That is, use the first FTC to evaluate $\int_1^x (4 - 2t) dt$.
- Observe that f is a linear function; what kind of function is A ?
- Using the formula you found in (b) that does not involve integrals, compute $A'(x)$.
- While we have defined f by the rule $f(t) = 4 - 2t$, it is equivalent to say that f is given by the rule $f(x) = 4 - 2x$. What do you observe about the relationship between A and f ?

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The Second Fundamental Theorem of Calculus

The result of Preview Activity 5.2 is not particular to the function $f(t) = 4 - 2t$, nor to the choice of “1” as the lower bound in the integral that defines the function A . For instance, if we let $f(t) = \cos(t) - t$ and set $A(x) = \int_2^x f(t) dt$, then we can determine a formula for A without integrals by the First FTC. Specifically,

$$\begin{aligned} A(x) &= \int_2^x (\cos(t) - t) dt \\ &= \sin(t) - \frac{1}{2}t^2 \Big|_2^x \\ &= \sin(x) - \frac{1}{2}x^2 - (\sin(2) - 2). \end{aligned}$$

Differentiating $A(x)$, since $(\sin(2) - 2)$ is constant, it follows that

$$A'(x) = \cos(x) - x,$$

and thus we see that $A'(x) = f(x)$. This tells us that for this particular choice of f , A is an antiderivative of f . More specifically, since $A(2) = \int_2^2 f(t) dt = 0$, A is the only antiderivative of f for which $A(2) = 0$.

In general, if f is any continuous function, and we define the function A by the rule

$$A(x) = \int_c^x f(t) dt,$$

where c is an arbitrary constant, then we can show that A is an antiderivative of f . To see why, let's demonstrate that $A'(x) = f(x)$ by using the limit definition of the derivative. Doing so, we observe that

$$\begin{aligned} A'(x) &= \lim_{h \rightarrow 0} \frac{A(x+h) - A(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{\int_c^{x+h} f(t) dt - \int_c^x f(t) dt}{h} \\ &= \lim_{h \rightarrow 0} \frac{\int_x^{x+h} f(t) dt}{h}, \end{aligned} \tag{5.3}$$

where Equation (5.3) in the preceding chain follows from the fact that $\int_c^x f(t) dt + \int_x^{x+h} f(t) dt = \int_c^{x+h} f(t) dt$. Now, observe that for small values of h ,

$$\int_x^{x+h} f(t) dt \approx f(x) \cdot h,$$

by a simple left-hand approximation of the integral. Thus, as we take the limit in Equation (5.3), it follows that

$$A'(x) = \lim_{h \rightarrow 0} \frac{\int_x^{x+h} f(t) dt}{h} = \lim_{h \rightarrow 0} \frac{f(x) \cdot h}{h} = f(x).$$

Hence, A is indeed an antiderivative of f . In addition, $A(c) = \int_c^c f(t) dt = 0$. The preceding argument demonstrates the truth of the Second Fundamental Theorem of Calculus, which we state as follows.

Theorem. (Second FTC) If f is a continuous function and c is any constant, then f has a unique antiderivative A that satisfies $A(c) = 0$, and that antiderivative is given by the rule $A(x) = \int_c^x f(t) dt$.

Activity 5.4.

Suppose that f is the function given in Figure 5.10 and that f is a piecewise function whose parts are either portions of lines or portions of circles, as pictured. In addition, let A be the function defined by the rule $A(x) = \int_2^x f(t) dt$.

- (a) What does the Second FTC tell us about the relationship between A and f ?
- (b) Compute $A(1)$ and $A(3)$ exactly.

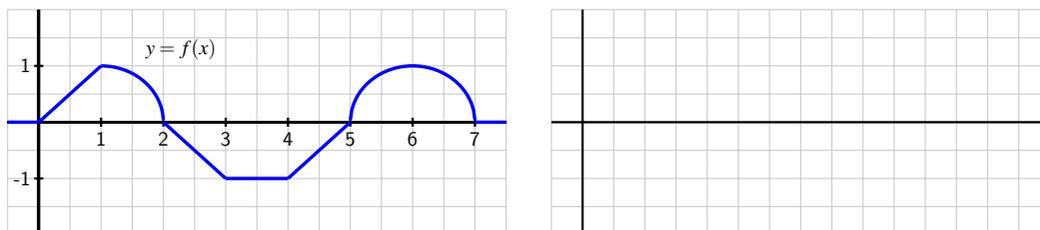


Figure 5.10: At left, the graph of $y = f(x)$. At right, axes for sketching $y = A(x)$.

- (c) Sketch a precise graph of $y = A(x)$ on the axes at right that accurately reflects where A is increasing and decreasing, where A is concave up and concave down, and the exact values of A at $x = 0, 1, \dots, 7$.
- (d) How is A similar to, but different from, the function F that you found in Activity 5.1?
- (e) With as little additional work as possible, sketch precise graphs of the functions $B(x) = \int_3^x f(t) dt$ and $C(x) = \int_1^x f(t) dt$. Justify your results with at least one sentence of explanation.

◁

Understanding Integral Functions

The Second FTC provides us with a means to construct an antiderivative of any continuous function. In particular, if we are given a continuous function g and wish to find an antiderivative of G , we can now say that

$$G(x) = \int_c^x g(t) dt$$

provides the rule for such an antiderivative, and moreover that $G(c) = 0$. Note especially that we know that $G'(x) = g(x)$. We sometimes want to write this relationship between G and g from a different notational perspective. In particular, observe that

$$\frac{d}{dx} \left[\int_c^x g(t) dt \right] = g(x). \quad (5.4)$$

This result can be particularly useful when we're given an integral function such as G and wish to understand properties of its graph by recognizing that $G'(x) = g(x)$, while not necessarily being able to exactly evaluate the definite integral $\int_c^x g(t) dt$. To see how this is the case, we consider the following example.

Example 5.1. Investigate the behavior of the integral function

$$E(x) = \int_0^x e^{-t^2} dt.$$

Solution. E is closely related to the well known *error function*², a function that is particularly important in probability and statistics. It turns out that the function e^{-t^2} does not have an elementary antiderivative that we can express without integrals. That is, whereas a function such as $f(t) = 4 - 2t$ has elementary antiderivative $F(t) = 4t - t^2$, we are unable to find a simple formula for an antiderivative of e^{-t^2} that does not involve a definite integral. We will learn more about finding (complicated) algebraic formulas for antiderivatives without definite integrals in the chapter on infinite series.

Returning our attention to the function E , while we cannot evaluate E exactly for any value other than $x = 0$, we still can gain a tremendous amount of information about the function E . To begin, applying the rule in Equation (5.4) to E , it follows that

$$E'(x) = \frac{d}{dx} \left[\int_0^x e^{-t^2} dt \right] = e^{-x^2},$$

so we know a formula for the derivative of E . Moreover, we know that $E(0) = 0$. This information is precisely the type we were given in problems such as the one in Activity 3.1 and others in Section 3.1, where we were given information about the derivative of a function, but lacked a formula for the function itself.

Here, using the first and second derivatives of E , along with the fact that $E(0) = 0$, we can determine more information about the behavior of E . First, with $E'(x) = e^{-x^2}$, we note that for all real numbers x , $e^{-x^2} > 0$, and thus $E'(x) > 0$ for all x . Thus E is an always increasing function. Further, we note that as $x \rightarrow \infty$, $E'(x) = e^{-x^2} \rightarrow 0$, hence the slope of the function E tends to zero as $x \rightarrow \infty$ (and similarly as $x \rightarrow -\infty$). Indeed, it turns out (due to some more sophisticated analysis) that E has horizontal asymptotes as x increases or decreases without bound.

In addition, we can observe that $E''(x) = -2xe^{-x^2}$, and that $E''(0) = 0$, while $E''(x) < 0$ for $x > 0$ and $E''(x) > 0$ for $x < 0$. This information tells us that E is concave up for $x < 0$ and concave down for $x > 0$ with a point of inflection at $x = 0$.

The only thing we lack at this point is a sense of how big E can get as x increases. If we use a midpoint Riemann sum with 10 subintervals to estimate $E(2)$, we see that $E(2) \approx 0.8822$; a similar calculation to estimate $E(3)$ shows little change ($E(3) \approx 0.8862$), so it appears that as x increases without bound, E approaches a value just larger than

²The error function is defined by the rule $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$ and has the key property that $0 \leq \operatorname{erf}(x) < 1$ for all $x \geq 0$ and moreover that $\lim_{x \rightarrow \infty} \operatorname{erf}(x) = 1$.

0.886, which aligns with the fact that E has horizontal asymptotes. Putting all of this information together (and using the symmetry of $f(t) = e^{-t^2}$), we see the results shown in Figure 5.11.

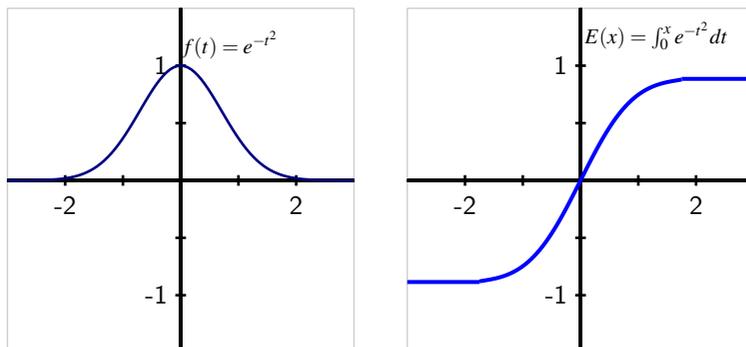


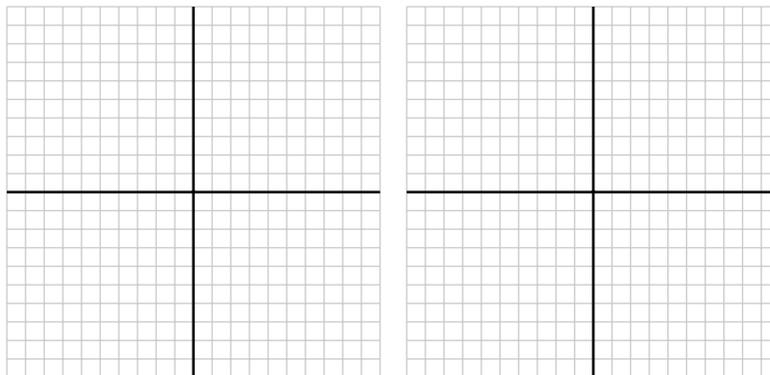
Figure 5.11: At left, the graph of $f(t) = e^{-t^2}$. At right, the integral function $E(x) = \int_0^x e^{-t^2} dt$, which is the unique antiderivative of f that satisfies $E(0) = 0$.

Again, E is the antiderivative of $f(t) = e^{-t^2}$ that satisfies $E(0) = 0$. Moreover, the values on the graph of $y = E(x)$ represent the net-signed area of the region bounded by $f(t) = e^{-t^2}$ from 0 up to x . We see that the value of E increases rapidly near zero but then levels off as x increases since there is less and less additional accumulated area bounded by $f(t) = e^{-t^2}$ as x increases.

Activity 5.5.

Suppose that $f(t) = \frac{t}{1+t^2}$ and $F(x) = \int_0^x f(t) dt$.

- On the axes at left in Figure 5.12, plot a graph of $f(t) = \frac{t}{1+t^2}$ on the interval $-10 \leq t \leq 10$. Clearly label the vertical axes with appropriate scale.
- What is the key relationship between F and f , according to the Second FTC?
- Use the first derivative test to determine the intervals on which F is increasing and decreasing.
- Use the second derivative test to determine the intervals on which F is concave up and concave down. Note that $f'(t)$ can be simplified to be written in the form $f'(t) = \frac{1-t^2}{(1+t^2)^2}$.
- Using technology appropriately, estimate the values of $F(5)$ and $F(10)$ through appropriate Riemann sums.

Figure 5.12: Axes for plotting f and F .

- (f) Sketch an accurate graph of $y = F(x)$ on the righthand axes provided, and clearly label the vertical axes with appropriate scale.

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Differentiating an Integral Function

We have seen that the Second FTC enables us to construct an antiderivative F of any continuous function f by defining F by the corresponding integral function $F(x) = \int_c^x f(t) dt$. Said differently, if we have a function of the form $F(x) = \int_c^x f(t) dt$, then we know that $F'(x) = \frac{d}{dx} \left[\int_c^x f(t) dt \right] = f(x)$. This shows that integral functions, while perhaps having the most complicated formulas of any functions we have encountered, are nonetheless particularly simple to differentiate. For instance, if

$$F(x) = \int_{\pi}^x \sin(t^2) dt,$$

then by the Second FTC, we know immediately that

$$F'(x) = \sin(x^2).$$

Stating this result more generally for an arbitrary function f , we know by the Second FTC that

$$\frac{d}{dx} \left[\int_a^x f(t) dt \right] = f(x).$$

In words, the last equation essentially says that “the derivative of the integral function whose integrand is f , is f .” In this sense, we see that if we first integrate the function f from $t = a$ to $t = x$, and then differentiate with respect to x , these two processes “undo”

one another.

Taking a different approach, say we begin with a function $f(t)$ and differentiate with respect to t . What happens if we follow this by integrating the result from $t = a$ to $t = x$? That is, what can we say about the quantity

$$\int_a^x \frac{d}{dt} [f(t)] dt?$$

Here, we use the First FTC and note that $f(t)$ is an antiderivative of $\frac{d}{dt} [f(t)]$. Applying this result and evaluating the antiderivative function, we see that

$$\begin{aligned} \int_a^x \frac{d}{dt} [f(t)] dt &= f(t) \Big|_a^x \\ &= f(x) - f(a). \end{aligned}$$

Thus, we see that if we apply the processes of first differentiating f and then integrating the result from a to x , we return to the function f , minus the constant value $f(a)$. So in this situation, the two processes almost undo one another, up to the constant $f(a)$.

The observations made in the preceding two paragraphs demonstrate that differentiating and integrating (where we integrate from a constant up to a variable) are almost inverse processes. In one sense, this should not be surprising: integrating involves antidifferentiating, which reverses the process of differentiating. On the other hand, we see that there is some subtlety involved, as integrating the derivative of a function does not quite produce the function itself. This is connected to a key fact we observed in Section 5.1, which is that any function has an entire family of antiderivatives, and any two of those antiderivatives differ only by a constant.

Activity 5.6.

Evaluate each of the following derivatives and definite integrals. Clearly cite whether you use the First or Second FTC in so doing.

(a) $\frac{d}{dx} \left[\int_4^x e^{t^2} dt \right]$

(b) $\int_{-2}^x \frac{d}{dt} \left[\frac{t^4}{1+t^4} \right] dt$

(c) $\frac{d}{dx} \left[\int_x^1 \cos(t^3) dt \right]$

(d) $\int_3^x \frac{d}{dt} [\ln(1+t^2)] dt$

(e) $\frac{d}{dx} \left[\int_4^{x^3} \sin(t^2) dt \right]$

(Hint: Let $F(x) = \int_4^x \sin(t^2) dt$ and observe that this problem is asking you to evaluate $\frac{d}{dx} [F(x^3)]$.)

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Summary

In this section, we encountered the following important ideas:

- For a continuous function f , the integral function $A(x) = \int_1^x f(t) dt$ defines an antiderivative of f .
- The Second Fundamental Theorem of Calculus is the formal, more general statement of the preceding fact: if f is a continuous function and c is any constant, then $A(x) = \int_c^x f(t) dt$ is the unique antiderivative of f that satisfies $A(c) = 0$.
- Together, the First and Second FTC enable us to formally see how differentiation and integration are almost inverse processes through the observations that

$$\int_c^x \frac{d}{dt} [f(t)] dt = f(x) - f(c)$$

and

$$\frac{d}{dx} \left[\int_c^x f(t) dt \right] = f(x).$$

Exercises

1. Let g be the function pictured at left in Figure 5.13, and let F be defined by $F(x) = \int_2^x g(t) dt$. Assume that the shaded areas have values $A_1 = 4.29$, $A_2 = 12.75$, $A_3 = 0.36$, and $A_4 = 1.79$. Assume further that the portion of A_2 that lies between $x = 0.5$ and $x = 2$ is 6.06.

Sketch a carefully labeled graph of F on the axes provided, and include a written analysis of how you know where F is zero, increasing, decreasing, CCU, and CCD.

2. The tide removes sand from the beach at a small ocean park at a rate modeled by the function

$$R(t) = 2 + 5 \sin\left(\frac{4\pi t}{25}\right)$$

A pumping station adds sand to the beach at rate modeled by the function

$$S(t) = \frac{15t}{1 + 3t}$$

Both $R(t)$ and $S(t)$ are measured in cubic yards of sand per hour, t is measured in hours, and the valid times are $0 \leq t \leq 6$. At time $t = 0$, the beach holds 2500 cubic yards of sand.

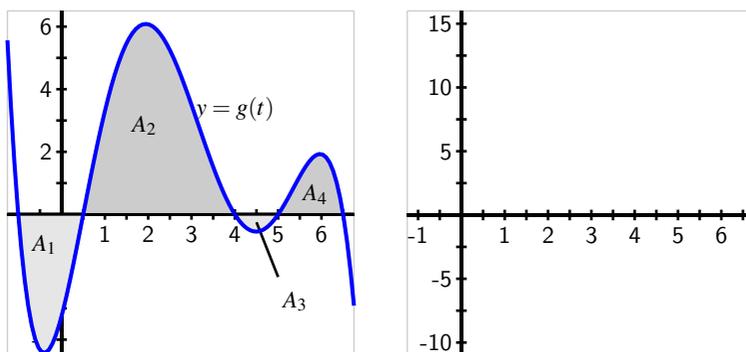


Figure 5.13: At left, the graph of g . At right, axes for plotting F .

- (a) What definite integral measures how much sand the tide will remove during the time period $0 \leq t \leq 6$? Why?
- (b) Write an expression for $Y(x)$, the total number of cubic yards of sand on the beach at time x . Carefully explain your thinking and reasoning.
- (c) At what instantaneous rate is the total number of cubic yards of sand on the beach at time $t = 4$ changing?
- (d) Over the time interval $0 \leq t \leq 6$, at what time t is the amount of sand on the beach least? What is this minimum value? Explain and justify your answers fully.
3. When an aircraft attempts to climb as rapidly as possible, its climb rate (in feet per minute) decreases as altitude increases, because the air is less dense at higher altitudes. Given below is a table showing performance data for a certain single engine aircraft, giving its climb rate at various altitudes, where $c(h)$ denotes the climb rate of the airplane at an altitude h .

h (feet)	0	1000	2000	3000	4000	5000	6000	7000	8000	9000	10,000
c (ft/min)	925	875	830	780	730	685	635	585	535	490	440

Let a new function m , that also depends on h , (say $y = m(h)$) measure the number of minutes required for a plane at altitude h to climb the next foot of altitude.

- a. Determine a similar table of values for $m(h)$ and explain how it is related to the table above. Be sure to discuss the units on m .
- b. Give a careful interpretation of a function whose derivative is $m(h)$. Describe what the input is and what the output is. Also, explain in plain English what the function tells us.

-
- c. Determine a definite integral whose value tells us exactly the number of minutes required for the airplane to ascend to 10,000 feet of altitude. Clearly explain why the value of this integral has the required meaning.
- d. Determine a formula for a function $M(h)$ whose value tells us the exact number of minutes required for the airplane to ascend to h feet of altitude.
- e. Estimate the values of $M(6000)$ and $M(10000)$ as accurately as you can. Include units on your results.
-

5.3 Integration by Substitution

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How can we begin to find algebraic formulas for antiderivatives of more complicated algebraic functions?
- What is an indefinite integral and how is its notation used in discussing antiderivatives?
- How does the technique of u -substitution work to help us evaluate certain indefinite integrals, and how does this process rely on identifying function-derivative pairs?

Introduction

In Section 4.4, we learned the key role that antiderivatives play in the process of evaluating definite integrals exactly. In particular, the Fundamental Theorem of Calculus tells us that if F is any antiderivative of f , then

$$\int_a^b f(x) dx = F(b) - F(a).$$

Furthermore, we realized that each elementary derivative rule developed in Chapter 2 leads to a corresponding elementary antiderivative, as summarized in Table 4.1. Thus, if we wish to evaluate an integral such as

$$\int_0^1 (x^3 - \sqrt{x} + 5^x) dx,$$

it is straightforward to do so, since we can easily antidifferentiate $f(x) = x^3 - \sqrt{x} + 5^x$. In particular, since a function F whose derivative is f is given by $F(x) = \frac{1}{4}x^4 - \frac{2}{3}x^{3/2} + \frac{1}{\ln(5)}5^x$, the Fundamental Theorem of Calculus tells us that

$$\begin{aligned} \int_0^1 (x^3 - \sqrt{x} + 5^x) dx &= \left. \frac{1}{4}x^4 - \frac{2}{3}x^{3/2} + \frac{1}{\ln(5)}5^x \right|_0^1 \\ &= \left(\frac{1}{4}(1)^4 - \frac{2}{3}(1)^{3/2} + \frac{1}{\ln(5)}5^1 \right) - \left(0 - 0 + \frac{1}{\ln(5)}5^0 \right) \\ &= -\frac{5}{12} + \frac{4}{\ln(5)}. \end{aligned}$$

Because an algebraic formula for an antiderivative of f enables us to evaluate the definite integral $\int_a^b f(x) dx$ exactly, we see that we have a natural interest in being able to find such algebraic antiderivatives. Note that we emphasize *algebraic* antiderivatives, as opposed to any antiderivative, since we know by the Second Fundamental Theorem of Calculus that $G(x) = \int_a^x f(t) dt$ is indeed an antiderivative of the given function f , but one that still involves a definite integral. One of our main goals in this section and the one following is to develop understanding, in select circumstances, of how to “undo” the process of differentiation in order to find an algebraic antiderivative for a given function.

Preview Activity 5.3. In Section 2.5, we learned the Chain Rule and how it can be applied to find the derivative of a composite function. In particular, if u is a differentiable function of x , and f is a differentiable function of $u(x)$, then

$$\frac{d}{dx} [f(u(x))] = f'(u(x)) \cdot u'(x).$$

In words, we say that the derivative of a composite function $c(x) = f(u(x))$, where f is considered the “outer” function and u the “inner” function, is “the derivative of the outer function, evaluated at the inner function, times the derivative of the inner function.”

- (a) For each of the following functions, use the Chain Rule to find the function’s derivative. Be sure to label each derivative by name (e.g., the derivative of $g(x)$ should be labeled $g'(x)$).

i. $g(x) = e^{3x}$

ii. $h(x) = \sin(5x + 1)$

iii. $p(x) = \arctan(2x)$

iv. $q(x) = (2 - 7x)^4$

v. $r(x) = 3^{4-11x}$

- (b) For each of the following functions, use your work in (a) to help you determine the general antiderivative³ of the function. Label each antiderivative by name (e.g., the antiderivative of m should be called M). In addition, check your work by computing the derivative of each proposed antiderivative.

i. $m(x) = e^{3x}$

ii. $n(x) = \cos(5x + 1)$

iii. $s(x) = \frac{1}{1+4x^2}$

³Recall that the general antiderivative of a function includes “+C” to reflect the entire family of functions that share the same derivative.

iv. $v(x) = (2 - 7x)^3$

v. $w(x) = 3^{4-11x}$

(c) Based on your experience in parts (a) and (b), conjecture an antiderivative for each of the following functions. Test your conjectures by computing the derivative of each proposed antiderivative.

i. $a(x) = \cos(\pi x)$

ii. $b(x) = (4x + 7)^{11}$

iii. $c(x) = xe^{x^2}$

∞

Reversing the Chain Rule: First Steps

In Preview Activity 5.3, we saw that it is usually straightforward to antidifferentiate a function of the form

$$h(x) = f(u(x)),$$

whenever f is a familiar function whose antiderivative is known and $u(x)$ is a linear function. For example, if we consider

$$h(x) = (5x - 3)^6,$$

in this context the outer function f is $f(u) = u^6$, while the inner function is $u(x) = 5x - 3$. Since the antiderivative of f is $F(u) = \frac{1}{7}u^7 + C$, we see that the antiderivative of h is

$$H(x) = \frac{1}{7}(5x - 3)^7 \cdot \frac{1}{5} + C = \frac{1}{35}(5x - 3)^7 + C.$$

The inclusion of the constant $\frac{1}{5}$ is essential precisely because the derivative of the inner function is $u'(x) = 5$. Indeed, if we now compute $H'(x)$, we find by the Chain Rule (and Constant Multiple Rule) that

$$H'(x) = \frac{1}{35} \cdot 7(5x - 3)^6 \cdot 5 = (5x - 3)^6 = h(x),$$

and thus H is indeed the general antiderivative of h .

Hence, in the special case where the outer function is familiar and the inner function

is linear, we can antidifferentiate composite functions according to the following rule.

If $h(x) = f(ax + b)$ and F is a known algebraic antiderivative of f , then the general antiderivative of h is given by

$$H(x) = \frac{1}{a}F(ax + b) + C.$$

When discussing antiderivatives, it is often useful to have shorthand notation that indicates the instruction to find an antiderivative. Thus, in a similar way to how the notation

$$\frac{d}{dx}[f(x)]$$

represents the derivative of $f(x)$ with respect to x , we use the notation of the *indefinite integral*,

$$\int f(x) dx$$

to represent the general antiderivative of f with respect to x . For instance, returning to the earlier example with $h(x) = (5x - 3)^6$ above, we can rephrase the relationship between h and its antiderivative H through the notation

$$\int (5x - 3)^6 dx = \frac{1}{35}(5x - 6)^7 + C.$$

When we find an antiderivative, we will often say that we *evaluate an indefinite integral*; said differently, the instruction to evaluate an indefinite integral means to find the general antiderivative. Just as the notation $\frac{d}{dx}[\square]$ means “find the derivative with respect to x of \square ,” the notation $\int \square dx$ means “find a function of x whose derivative is \square .”

Activity 5.7.

Evaluate each of the following indefinite integrals. Check each antiderivative that you find by differentiating.

(a) $\int \sin(8 - 3x) dx$

(b) $\int \sec^2(4x) dx$

(c) $\int \frac{1}{11x-9} dx$

(d) $\int \csc(2x + 1) \cot(2x + 1) dx$

(e) $\int \frac{1}{\sqrt{1-16x^2}} dx$

(f) $\int 5^{-x} dx$

Reversing the Chain Rule: u -substitution

Of course, a natural question arises from our recent work: what happens when the inner function is not a linear function? For example, can we find antiderivatives of such functions as

$$g(x) = xe^{x^2} \text{ and } h(x) = e^{x^2}?$$

It is important to explicitly remember that differentiation and antidifferentiation are essentially inverse processes; that they are not quite inverse processes is due to the $+C$ that arises when antidifferentiating. This close relationship enables us to take any known derivative rule and translate it to a corresponding rule for an indefinite integral. For example, since

$$\frac{d}{dx} [x^5] = 5x^4,$$

we can equivalently write

$$\int 5x^4 dx = x^5 + C.$$

Recall that the Chain Rule states that

$$\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x).$$

Restating this relationship in terms of an indefinite integral,

$$\int f'(g(x))g'(x) dx = f(g(x)) + C. \quad (5.5)$$

Hence, Equation (5.5) tells us that if we can take a given function and view its algebraic structure as $f'(g(x))g'(x)$ for some appropriate choices of f and g , then we can antidifferentiate the function by reversing the Chain Rule. It is especially notable that both $g(x)$ and $g'(x)$ appear in the form of $f'(g(x))g'(x)$; we will sometimes say that we seek to *identify a function-derivative pair* when trying to apply the rule in Equation (5.5).

In the situation where we can identify a function-derivative pair, we will introduce a new variable u to represent the function $g(x)$. Observing that with $u = g(x)$, it follows in Leibniz notation that $\frac{du}{dx} = g'(x)$, so that in terms of differentials⁴, $du = g'(x) dx$. Now converting the indefinite integral of interest to a new one in terms of u , we have

$$\int f'(g(x))g'(x) dx = \int f'(u) du.$$

Provided that f' is an elementary function whose antiderivative is known, we can now

⁴If we recall from the definition of the derivative that $\frac{du}{dx} \approx \frac{\Delta u}{\Delta x}$ and use the fact that $\frac{du}{dx} = g'(x)$, then we see that $g'(x) \approx \frac{\Delta u}{\Delta x}$. Solving for Δu , $\Delta u \approx g'(x)\Delta x$. It is this last relationship that, when expressed in "differential" notation enables us to write $du = g'(x) dx$ in the change of variable formula.

easily evaluate the indefinite integral in u , and then go on to determine the desired overall antiderivative of $f'(g(x))g'(x)$. We call this process u -substitution. To see u -substitution at work, we consider the following example.

Example 5.2. Evaluate the indefinite integral

$$\int x^3 \cdot \sin(7x^4 + 3) dx$$

and check the result by differentiating.

Solution. We can make two key algebraic observations regarding the integrand, $x^3 \cdot \sin(7x^4 + 3)$. First, $\sin(7x^4 + 3)$ is a composite function; as such, we know we'll need a more sophisticated approach to antidifferentiating. Second, x^3 is almost the derivative of $(7x^4 + 3)$; the only issue is a missing constant. Thus, x^3 and $(7x^4 + 3)$ are nearly a function-derivative pair. Furthermore, we know the antiderivative of $f(u) = \sin(u)$. The combination of these observations suggests that we can evaluate the given indefinite integral by reversing the chain rule through u -substitution.

Letting u represent the inner function of the composite function $\sin(7x^4 + 3)$, we have $u = 7x^4 + 3$, and thus $\frac{du}{dx} = 28x^3$. In differential notation, it follows that $du = 28x^3 dx$, and thus $x^3 dx = \frac{1}{28} du$. We make this last observation because the original indefinite integral may now be written

$$\int \sin(7x^4 + 3) \cdot x^3 dx,$$

and so by substituting the expressions in u for x (specifically u for $7x^4 + 3$ and $\frac{1}{28} du$ for $x^3 dx$), it follows that

$$\int \sin(7x^4 + 3) \cdot x^3 dx = \int \sin(u) \cdot \frac{1}{28} du.$$

Now we may evaluate the original integral by first evaluating the easier integral in u , followed by replacing u by the expression $7x^4 + 3$. Doing so, we find

$$\begin{aligned} \int \sin(7x^4 + 3) \cdot x^3 dx &= \int \sin(u) \cdot \frac{1}{28} du \\ &= \frac{1}{28} \int \sin(u) du \\ &= \frac{1}{28} (-\cos(u)) + C \\ &= -\frac{1}{28} \cos(7x^4 + 3) + C. \end{aligned}$$

To check our work, we observe by the Chain Rule that

$$\frac{d}{dx} \left[-\frac{1}{28} \cos(7x^4 + 3) + C \right] = -\frac{1}{28} \cdot (-1) \sin(7x^4 + 3) \cdot 28x^3 = \sin(7x^4 + 3) \cdot x^3,$$

which is indeed the original integrand.

An essential observation about our work in Example 5.2 is that the u -substitution only worked because the function multiplying $\sin(7x^4 + 3)$ was x^3 . If instead that function was x^2 or x^4 , the substitution process may not (and likely would not) have worked. This is one of the primary challenges of antidifferentiation: slight changes in the integrand make tremendous differences. For instance, we can use u -substitution with $u = x^2$ and $du = 2x dx$ to find that

$$\begin{aligned} \int x e^{x^2} dx &= \int e^u \cdot \frac{1}{2} du \\ &= \frac{1}{2} \int e^u du \\ &= \frac{1}{2} e^u + C \\ &= \frac{1}{2} e^{x^2} + C. \end{aligned}$$

If, however, we consider the similar indefinite integral

$$\int e^{x^2} dx,$$

the missing x to multiply e^{x^2} makes the u -substitution $u = x^2$ no longer possible. Hence, part of the lesson of u -substitution is just how specialized the process is: it only applies to situations where, up to a missing constant, the integrand that is present is the result of applying the Chain Rule to a different, related function.

Activity 5.8.

Evaluate each of the following indefinite integrals by using these steps:

- Find two functions within the integrand that form (up to a possible missing constant) a function-derivative pair;
- Make a substitution and convert the integral to one involving u and du ;
- Evaluate the new integral in u ;
- Convert the resulting function of u back to a function of x by using your earlier substitution;

- Check your work by differentiating the function of x . You should come up with the integrand originally given.

$$(a) \int \frac{x^2}{5x^3 + 1} dx$$

$$(b) \int e^x \sin(e^x) dx$$

$$(c) \int \frac{\cos(\sqrt{x})}{\sqrt{x}} dx$$

◁

Evaluating Definite Integrals via u -substitution

We have just introduced u -substitution as a means to evaluate indefinite integrals of functions that can be written, up to a constant multiple, in the form $f(g(x))g'(x)$. This same technique can be used to evaluate definite integrals involving such functions, though we need to be careful with the corresponding limits of integration. Consider, for instance, the definite integral

$$\int_2^5 xe^{x^2} dx.$$

Whenever we write a definite integral, it is implicit that the limits of integration correspond to the variable of integration. To be more explicit, observe that

$$\int_2^5 xe^{x^2} dx = \int_{x=2}^{x=5} xe^{x^2} dx.$$

When we execute a u -substitution, we change the *variable* of integration; it is essential to note that this also changes the *limits* of integration. For instance, with the substitution $u = x^2$ and $du = 2x dx$, it also follows that when $x = 2$, $u = 2^2 = 4$, and when $x = 5$, $u = 5^2 = 25$. Thus, under the change of variables of u -substitution, we now have

$$\begin{aligned} \int_{x=2}^{x=5} xe^{x^2} dx &= \int_{u=4}^{u=25} e^u \cdot \frac{1}{2} du \\ &= \frac{1}{2} e^u \Big|_{u=4}^{u=25} \\ &= \frac{1}{2} e^{25} - \frac{1}{2} e^4. \end{aligned}$$

Alternatively, we could consider the related indefinite integral $\int xe^{x^2} dx$, find the antiderivative $\frac{1}{2}e^{x^2}$ through u -substitution, and then evaluate the original definite integral.

From that perspective, we'd have

$$\begin{aligned}\int_2^5 x e^{x^2} dx &= \left. \frac{1}{2} e^{x^2} \right|_2^5 \\ &= \frac{1}{2} e^{25} - \frac{1}{2} e^4,\end{aligned}$$

which is, of course, the same result.

Activity 5.9.

Evaluate each of the following definite integrals exactly through an appropriate u -substitution.

(a) $\int_1^2 \frac{x}{1+4x^2} dx$

(b) $\int_0^1 e^{-x}(2e^{-x} + 3)^9 dx$

(c) $\int_{2/\pi}^{4/\pi} \frac{\cos(\frac{1}{x})}{x^2} dx$

◁

Summary

In this section, we encountered the following important ideas:

- To begin to find algebraic formulas for antiderivatives of more complicated algebraic functions, we need to think carefully about how we can reverse known differentiation rules. To that end, it is essential that we understand and recall known derivatives of basic functions, as well as the standard derivative rules.
- The indefinite integral provides notation for antiderivatives. When we write “ $\int f(x) dx$,” we mean “the general antiderivative of f .” In particular, if we have functions f and F such that $F' = f$, the following two statements say the exact thing:

$$\frac{d}{dx}[F(x)] = f(x) \text{ and } \int f(x) dx = F(x) + C.$$

That is, f is the derivative of F , and F is an antiderivative of f .

- The technique of u -substitution helps us evaluate indefinite integrals of the form $\int f(g(x))g'(x) dx$ through the substitutions $u = g(x)$ and $du = g'(x) dx$, so that

$$\int f(g(x))g'(x) dx = \int f(u) du.$$

A key part of choosing the expression in x to be represented by u is the identification of a function-derivative pair. To do so, we often look for an “inner” function $g(x)$ that is part of a composite function, while investigating whether $g'(x)$ (or a constant multiple of $g'(x)$) is present as a multiplying factor of the integrand.

Exercises

1. This problem centers on finding antiderivatives for the basic trigonometric functions other than $\sin(x)$ and $\cos(x)$.

(a) Consider the indefinite integral $\int \tan(x) dx$. By rewriting the integrand as $\tan(x) = \frac{\sin(x)}{\cos(x)}$ and identifying an appropriate function-derivative pair, make a u -substitution and hence evaluate $\int \tan(x) dx$.

(b) In a similar way, evaluate $\int \cot(x) dx$.

(c) Consider the indefinite integral

$$\int \frac{\sec^2(x) + \sec(x) \tan(x)}{\sec(x) + \tan(x)} dx.$$

Evaluate this integral using the substitution $u = \sec(x) + \tan(x)$.

(d) Simplify the integrand in (c) by factoring the numerator. What is a far simpler way to write the integrand?

(e) Combine your work in (c) and (d) to determine $\int \sec(x) dx$.

(f) Using (c)-(e) as a guide, evaluate $\int \csc(x) dx$.

2. Consider the indefinite integral $\int x\sqrt{x-1} dx$.

(a) At first glance, this integrand may not seem suited to substitution due to the presence of x in separate locations in the integrand. Nonetheless, using the composite function $\sqrt{x-1}$ as a guide, let $u = x-1$. Determine expressions for both x and dx in terms of u .

(b) Convert the given integral in x to a new integral in u .

(c) Evaluate the integral in (b) by noting that $\sqrt{u} = u^{1/2}$ and observing that it is now possible to rewrite the integrand in u by expanding through multiplication.

(d) Evaluate each of the integrals $\int x^2\sqrt{x-1} dx$ and $\int x\sqrt{x^2-1} dx$. Write a

paragraph to discuss the similarities among the three indefinite integrals in this problem and the role of substitution and algebraic rearrangement in each.

3. Consider the indefinite integral $\int \sin^3(x) dx$.
- Explain why the substitution $u = \sin(x)$ will not work to help evaluate the given integral.
 - Recall the Fundamental Trigonometric Identity, which states that $\sin^2(x) + \cos^2(x) = 1$. By observing that $\sin^3(x) = \sin(x) \cdot \sin^2(x)$, use the Fundamental Trigonometric Identity to rewrite the integrand as the product of $\sin(x)$ with another function.
 - Explain why the substitution $u = \cos(x)$ now provides a possible way to evaluate the integral in (b).
 - Use your work in (a)-(c) to evaluate the indefinite integral $\int \sin^3(x) dx$.
 - Use a similar approach to evaluate $\int \cos^3(x) dx$.
4. For the town of Mathland, MI, residential power consumption has shown certain trends over recent years. Based on data reflecting average usage, engineers at the power company have modeled the town's rate of energy consumption by the function

$$r(t) = 4 + \sin(0.263t + 4.7) + \cos(0.526t + 9.4).$$

Here, t measures time in hours after midnight on a typical weekday, and r is the rate of consumption in megawatts⁵ at time t . Units are critical throughout this problem.

- Sketch a carefully labeled graph of $r(t)$ on the interval $[0,24]$ and explain its meaning. Why is this a reasonable model of power consumption?
- Without calculating its value, explain the meaning of $\int_0^{24} r(t) dt$. Include appropriate units on your answer.
- Determine the exact amount of power Mathland consumes in a typical day.
- What is Mathland's average rate of energy consumption in a given 24-hour period? What are the units on this quantity?

⁵The unit *megawatt* is itself a rate, which measures energy consumption per unit time. A *megawatt-hour* is the total amount of energy that is equivalent to a constant stream of 1 megawatt of power being sustained for 1 hour.

5.4 Integration by Parts

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How do we evaluate indefinite integrals that involve products of basic functions such as $\int x \sin(x) dx$ and $\int xe^x dx$?
- What is the method of integration by parts and how can we consistently apply it to integrate products of basic functions?
- How does the algebraic structure of functions guide us in identifying u and dv in using integration by parts?

Introduction

In Section 5.3, we learned the technique of u -substitution for evaluating indefinite integrals that involve certain composite functions. For example, the indefinite integral $\int x^3 \sin(x^4) dx$ is perfectly suited to u -substitution, since not only is there a composite function present, but also the inner function's derivative (up to a constant) is multiplying the composite function. Through u -substitution, we learned a general situation where recognizing the algebraic structure of a function can enable us to find its antiderivative.

It is natural to ask similar questions to those we considered in Section 5.3 about functions with a different elementary algebraic structure: those that are the product of basic functions. For instance, suppose we are interested in evaluating the indefinite integral

$$\int x \sin(x) dx.$$

Here, there is not a composite function present, but rather a product of the basic functions $f(x) = x$ and $g(x) = \sin(x)$. From our work in Section 2.3 with the Product Rule, we know that it is relatively complicated to compute the derivative of the product of two functions, so we should expect that antidifferentiating a product should be similarly involved. In addition, intuitively we expect that evaluating $\int x \sin(x) dx$ will involve somehow reversing the Product Rule.

To that end, in Preview Activity 5.4 we refresh our understanding of the Product Rule and then investigate some indefinite integrals that involve products of basic functions.

Preview Activity 5.4. In Section 2.3, we developed the Product Rule and studied how it is employed to differentiate a product of two functions. In particular, recall that if f and g

are differentiable functions of x , then

$$\frac{d}{dx} [f(x) \cdot g(x)] = f(x) \cdot g'(x) + g(x) \cdot f'(x).$$

(a) For each of the following functions, use the Product Rule to find the function's derivative. Be sure to label each derivative by name (e.g., the derivative of $g(x)$ should be labeled $g'(x)$).

i. $g(x) = x \sin(x)$

ii. $h(x) = xe^x$

iii. $p(x) = x \ln(x)$

iv. $q(x) = x^2 \cos(x)$

v. $r(x) = e^x \sin(x)$

(b) Use your work in (a) to help you evaluate the following indefinite integrals. Use differentiation to check your work.

i. $\int xe^x + e^x dx$

ii. $\int e^x(\sin(x) + \cos(x)) dx$

iii. $\int 2x \cos(x) - x^2 \sin(x) dx$

iv. $\int x \cos(x) + \sin(x) dx$

v. $\int 1 + \ln(x) dx$

(c) Observe that the examples in (b) work nicely because of the derivatives you were asked to calculate in (a). Each integrand in (b) is precisely the result of differentiating one of the products of basic functions found in (a). To see what happens when an integrand is still a product but not necessarily the result of differentiating an elementary product, we consider how to evaluate

$$\int x \cos(x) dx.$$

i. First, observe that

$$\frac{d}{dx} [x \sin(x)] = x \cos(x) + \sin(x).$$

Integrating both sides indefinitely and using the fact that the integral of a sum is the sum of the integrals, we find that

$$\int \left(\frac{d}{dx} [x \sin(x)] \right) dx = \int x \cos(x) dx + \int \sin(x) dx.$$

In this last equation, evaluate the indefinite integral on the left side as well as the rightmost indefinite integral on the right.

- ii. In the most recent equation from (i), solve the equation for the expression $\int x \cos(x) dx$.
- iii. For which product of basic functions have you now found the antiderivative?

⌘

Reversing the Product Rule: Integration by Parts

Problem (c) in Preview Activity 5.4 provides a clue for how we develop the general technique known as Integration by Parts, which comes from reversing the Product Rule. Recall that the Product Rule states that

$$\frac{d}{dx} [f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

Integrating both sides of this equation indefinitely with respect to x , it follows that

$$\int \frac{d}{dx} [f(x)g(x)] dx = \int f(x)g'(x) dx + \int g(x)f'(x) dx. \quad (5.6)$$

On the left in Equation (5.6), we recognize that we have the indefinite integral of the derivative of a function which, up to an additional constant, is the original function itself. Temporarily omitting the constant that may arise, we equivalently have

$$f(x)g(x) = \int f(x)g'(x) dx + \int g(x)f'(x) dx. \quad (5.7)$$

The most important thing to observe about Equation (5.7) is that it provides us with a choice of two integrals to evaluate. That is, in a situation where we can identify two functions f and g , if we can integrate $f(x)g'(x)$, then we know the indefinite integral of $g(x)f'(x)$, and vice versa. To that end, we choose the first indefinite integral on the left in

Equation (5.7) and solve for it to generate the rule

$$\int f(x)g'(x) dx = f(x)g(x) - \int g(x)f'(x) dx. \quad (5.8)$$

Often we express Equation (5.8) in terms of the variables u and v , where $u = f(x)$ and $v = g(x)$. Note that in differential notation, $du = f'(x) dx$ and $dv = g'(x) dx$, and thus we can state the rule for Integration by Parts in its most common form as follows.

$$\int u dv = uv - \int v du.$$

To apply Integration by Parts, we look for a product of basic functions that we can identify as u and dv . If we can antidifferentiate dv to find v , and evaluating $\int v du$ is not more difficult than evaluating $\int u dv$, then this substitution usually proves to be fruitful. To demonstrate, we consider the following example.

Example 5.3. Evaluate the indefinite integral

$$\int x \cos(x) dx$$

using Integration by Parts.

Solution. Whenever we are trying to integrate a product of basic functions through Integration by Parts, we are presented with a choice for u and dv . In the current problem, we can either let $u = x$ and $dv = \cos(x) dx$, or let $u = \cos(x)$ and $dv = x dx$. While there is not a universal rule for how to choose u and dv , a good guideline is this: do so in a way that $\int v du$ is at least as simple as the original problem $\int u dv$.

In this setting, this leads us to choose⁶ $u = x$ and $dv = \cos(x) dx$, from which it follows that $du = 1 dx$ and $v = \sin(x)$. With this substitution, the rule for Integration by Parts tells us that

$$\int x \cos(x) dx = x \sin(x) - \int \sin(x) \cdot 1 dx.$$

⁶Observe that if we considered the alternate choice, and let $u = \cos(x)$ and $dv = x dx$, then $du = -\sin(x) dx$ and $v = \frac{1}{2}x^2$, from which we would write

$$\int x \cos(x) dx = \frac{1}{2}x^2 \cos(x) - \int \frac{1}{2}x^2(-\sin(x)) dx.$$

Thus we have replaced the problem of integrating $x \cos(x)$ with that of integrating $\frac{1}{2}x^2 \sin(x)$; the latter is clearly more complicated, which shows that this alternate choice is not as helpful as the first choice.

At this point, all that remains to do is evaluate the (simpler) integral $\int \sin(x) \cdot 1 \, dx$. Doing so, we find

$$\int x \cos(x) \, dx = x \sin(x) - (-\cos(x)) + C = x \sin(x) + \cos(x) + C.$$

There are at least two additional important observations to make from Example 5.3. First, the general technique of Integration by Parts involves trading the problem of integrating the product of two functions for the problem of integrating the product of two related functions. In particular, we convert the problem of evaluating $\int u \, dv$ for that of evaluating $\int v \, du$. This perspective clearly shapes our choice of u and v . In Example 5.3, the original integral to evaluate was $\int x \cos(x) \, dx$, and through the substitution provided by Integration by Parts, we were instead able to evaluate $\int \sin(x) \cdot 1 \, dx$. Note that the original function x was replaced by its derivative, while $\cos(x)$ was replaced by its antiderivative. Second, observe that when we get to the final stage of evaluating the last remaining antiderivative, it is at this step that we include the integration constant, $+C$.

Activity 5.10.

Evaluate each of the following indefinite integrals. Check each antiderivative that you find by differentiating.

(a) $\int t e^{-t} \, dt$

(b) $\int 4x \sin(3x) \, dx$

(c) $\int z \sec^2(z) \, dz$

(d) $\int x \ln(x) \, dx$

◀

Some Subtleties with Integration by Parts

There are situations where Integration by Parts is not an obvious choice, but the technique is appropriate nonetheless. One guide to understanding why is the observation that integration by parts allows us to replace one function in a product with its derivative while replacing the other with its antiderivative. For instance, consider the problem of evaluating

$$\int \arctan(x) \, dx.$$

Initially, this problem seems ill-suited to Integration by Parts, since there does not appear to be a product of functions present. But if we note that $\arctan(x) = \arctan(x) \cdot 1$, and realize that we know the derivative of $\arctan(x)$ as well as the antiderivative of 1, we

see the possibility for the substitution $u = \arctan(x)$ and $dv = 1 dx$. We explore this substitution further in Activity 5.11.

In a related problem, if we consider $\int t^3 \sin(t^2) dt$, two key observations can be made about the algebraic structure of the integrand: there is a composite function present in $\sin(t^2)$, and there is not an obvious function-derivative pair, as we have t^3 present (rather than simply t) multiplying $\sin(t^2)$. This problem exemplifies the situation where we sometimes use both u -substitution and Integration by Parts in a single problem. If we write $t^3 = t \cdot t^2$ and consider the indefinite integral

$$\int t \cdot t^2 \cdot \sin(t^2) dt,$$

we can use a mix of the two techniques we have recently learned. First, let $z = t^2$ so that $dz = 2t dt$, and thus $t dt = \frac{1}{2} dz$. (We are using the variable z to perform a “ z -substitution” since u will be used subsequently in executing Integration by Parts.) Under this z -substitution, we now have

$$\int t \cdot t^2 \cdot \sin(t^2) dt = \int z \cdot \sin(z) \cdot \frac{1}{2} dz.$$

The remaining integral is a standard one that can be evaluated by parts. This, too, is explored further in Activity 5.11.

The problems briefly introduced here exemplify that we sometimes must think creatively in choosing the variables for substitution in Integration by Parts, as well as that it is entirely possible that we will need to use the technique of substitution for an additional change of variables within the process of integrating by parts.

Activity 5.11.

Evaluate each of the following indefinite integrals, using the provided hints.

- Evaluate $\int \arctan(x) dx$ by using Integration by Parts with the substitution $u = \arctan(x)$ and $dv = 1 dx$.
- Evaluate $\int \ln(z) dz$. Consider a similar substitution to the one in (a).
- Use the substitution $z = t^2$ to transform the integral $\int t^3 \sin(t^2) dt$ to a new integral in the variable z , and evaluate that new integral by parts.
- Evaluate $\int s^5 e^{s^3} ds$ using an approach similar to that described in (c).
- Evaluate $\int e^{2t} \cos(e^t) dt$. You will find it helpful to note that $e^{2t} = e^t \cdot e^t$.

◀

Using Integration by Parts Multiple Times

We have seen that the technique of Integration by Parts is well suited to integrating the product of basic functions, and that it allows us to essentially trade a given integrand for a new one where one function in the product is replaced by its derivative, while the other is replaced by its antiderivative. The main goal in this trade of $\int u dv$ for $\int v du$ is to have the new integral not be more challenging to evaluate than the original one. At times, it turns out that it can be necessary to apply Integration by Parts more than once in order to ultimately evaluate a given indefinite integral.

For example, if we consider $\int t^2 e^t dt$ and let $u = t^2$ and $dv = e^t dt$, then it follows that $du = 2t dt$ and $v = e^t$, thus

$$\int t^2 e^t dt = t^2 e^t - \int 2te^t dt.$$

The integral on the righthand side is simpler to evaluate than the one on the left, but it still requires Integration by Parts. Now letting $u = 2t$ and $dv = e^t dt$, we have $du = 2 dt$ and $v = e^t$, so that

$$\int t^2 e^t dt = t^2 e^t - \left(2te^t - \int 2e^t dt \right).$$

Note the key role of the parentheses, as it is essential to distribute the minus sign to the entire value of the integral $\int 2te^t dt$. The final integral on the right in the most recent equation is a basic one; evaluating that integral and distributing the minus sign, we find

$$\int t^2 e^t dt = t^2 e^t - 2te^t + 2e^t + C.$$

Of course, situations are possible where even more than two applications of Integration by Parts may be necessary. For instance, in the preceding example, it is apparent that if the integrand was $t^3 e^t$ instead, we would have to use Integration by Parts three times.

Next, we consider the slightly different scenario presented by the definite integral $\int e^t \cos(t) dt$. Here, we can choose to let u be either e^t or $\cos(t)$; we pick $u = \cos(t)$, and thus $dv = e^t dt$. With $du = -\sin(t) dt$ and $v = e^t$, Integration by Parts tells us that

$$\int e^t \cos(t) dt = e^t \cos(t) - \int e^t (-\sin(t)) dt,$$

or equivalently that

$$\int e^t \cos(t) dt = e^t \cos(t) + \int e^t \sin(t) dt \tag{5.9}$$

Observe that the integral on the right in Equation (5.9), $\int e^t \sin(t) dt$, while not being more complicated than the original integral we want to evaluate, it is essentially identical

to $\int e^t \cos(t) dt$. While the overall situation isn't necessarily better than what we started with, the problem hasn't gotten worse. Thus, we proceed by integrating by parts again. This time we let $u = \sin(t)$ and $dv = e^t dt$, so that $du = \cos(t) dt$ and $v = e^t$, which implies

$$\int e^t \cos(t) dt = e^t \cos(t) + \left(e^t \sin(t) - \int e^t \cos(t) dt \right) \quad (5.10)$$

We seem to be back where we started, as two applications of Integration by Parts has led us back to the original problem, $\int e^t \cos(t) dt$. But if we look closely at Equation (5.10), we see that we can use algebra to solve for the value of the desired integral. In particular, adding $\int e^t \cos(t) dt$ to both sides of the equation, we have

$$2 \int e^t \cos(t) dt = e^t \cos(t) + e^t \sin(t),$$

and therefore

$$\int e^t \cos(t) dt = \frac{1}{2} (e^t \cos(t) + e^t \sin(t)) + C.$$

Note that since we never actually encountered an integral we could evaluate directly, we didn't have the opportunity to add the integration constant C until the final step, at which point we include it as part of the most general antiderivative that we sought from the outset in evaluating an indefinite integral.

Activity 5.12.

Evaluate each of the following indefinite integrals.

(a) $\int x^2 \sin(x) dx$

(b) $\int t^3 \ln(t) dt$

(c) $\int e^z \sin(z) dz$

(d) $\int s^2 e^{3s} ds$

(e) $\int t \arctan(t) dt$

(Hint: At a certain point in this problem, it is very helpful to note that $\frac{t^2}{1+t^2} = 1 - \frac{1}{1+t^2}$.)

Evaluating Definite Integrals Using Integration by Parts

Just as we saw with u -substitution in Section 5.3, we can use the technique of Integration by Parts to evaluate a definite integral. Say, for example, we wish to find the exact value of

$$\int_0^{\pi/2} t \sin(t) dt.$$

One option is to evaluate the related indefinite integral to find that $\int t \sin(t) dt = -t \cos(t) + \sin(t) + C$, and then use the resulting antiderivative along with the Fundamental Theorem of Calculus to find that

$$\begin{aligned} \int_0^{\pi/2} t \sin(t) dt &= (-t \cos(t) + \sin(t)) \Big|_0^{\pi/2} \\ &= \left(-\frac{\pi}{2} \cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) \right) - (-0 \cos(0) + \sin(0)) \\ &= 1. \end{aligned}$$

Alternatively, we can apply Integration by Parts and work with definite integrals throughout. In this perspective, it is essential to remember to evaluate the product uv over the given limits of integration. To that end, using the substitution $u = t$ and $dv = \sin(t) dt$, so that $du = dt$ and $v = -\cos(t)$, we write

$$\begin{aligned} \int_0^{\pi/2} t \sin(t) dt &= -t \cos(t) \Big|_0^{\pi/2} - \int_0^{\pi/2} (-\cos(t)) dt \\ &= -t \cos(t) \Big|_0^{\pi/2} + \sin(t) \Big|_0^{\pi/2} \\ &= \left(-\frac{\pi}{2} \cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{\pi}{2}\right) \right) - (-0 \cos(0) + \sin(0)) \\ &= 1. \end{aligned}$$

As with any substitution technique, it is important to remember the overall goal of the problem, to use notation carefully and completely, and to think about our end result to ensure that it makes sense in the context of the question being answered.

When u -substitution and Integration by Parts Fail to Help

As we close this section, it is important to note that both integration techniques we have discussed apply in relatively limited circumstances. In particular, it is not hard to find examples of functions for which neither technique produces an antiderivative; indeed, there are many, many functions that appear elementary but that do not have an elementary

algebraic antiderivative. For instance, if we consider the indefinite integrals

$$\int e^{x^2} dx \quad \text{and} \quad \int x \tan(x) dx,$$

neither u -substitution nor Integration by Parts proves fruitful. While there are other integration techniques, some of which we will consider briefly, none of them enables us to find an algebraic antiderivative for e^{x^2} or $x \tan(x)$. There are at least two key observations to make: one, we do know from the Second Fundamental Theorem of Calculus that we can construct an integral antiderivative for each function; and two, antidifferentiation is much, much harder in general than differentiation. In particular, we observe that $F(x) = \int_0^x e^{t^2} dt$ is an antiderivative of $f(x) = e^{x^2}$, and $G(x) = \int_0^x t \tan(t) dt$ is an antiderivative of $g(x) = x \tan(x)$. But finding an elementary algebraic formula that doesn't involve integrals for either F or G turns out not only to be impossible through u -substitution or Integration by Parts, but indeed impossible altogether.

Summary

In this section, we encountered the following important ideas:

- Through the method of Integration by Parts, we can evaluate indefinite integrals that involve products of basic functions such as $\int x \sin(x) dx$ and $\int x \ln(x) dx$ through a substitution that enables us to effectively trade one of the functions in the product for its derivative, and the other for its antiderivative, in an effort to find a different product of functions that is easier to integrate.
- If we are given an integral whose algebraic structure we can identify as a product of basic functions in the form $\int f(x)g'(x) dx$, we can use the substitution $u = f(x)$ and $dv = g'(x) dx$ and apply the rule

$$\int u dv = uv - \int v du$$

in an effort to evaluate the original integral $\int f(x)g'(x) dx$ by instead evaluating $\int v du = \int f'(x)g(x) dx$.

- When deciding to integrate by parts, we normally have a product of functions present in the integrand and we have to select both u and dv . That selection is guided by the overall principal that we desire the new integral $\int v du$ to not be any more difficult or complicated than the original integral $\int u dv$. In addition, it is often helpful to recognize if one of the functions present is much easier to differentiate than antidifferentiate (such as $\ln(x)$), in which case that function often is best assigned the variable u . For sure, when choosing dv , the corresponding function must be one that we can antidifferentiate.
-

Exercises

1. Let $f(t) = te^{-2t}$ and $F(x) = \int_0^x f(t) dt$.
 - (a) Determine $F'(x)$.
 - (b) Use the First FTC to find a formula for F that does not involve an integral.
 - (c) Is F an increasing or decreasing function for $x > 0$? Why?
2. Consider the indefinite integral given by $\int e^{2x} \cos(e^x) dx$.
 - (a) Noting that $e^{2x} = e^x \cdot e^x$, use the substitution $z = e^x$ to determine a new, equivalent integral in the variable z .
 - (b) Evaluate the integral you found in (a) using an appropriate technique.
 - (c) How is the problem of evaluating $\int e^{2x} \cos(e^{2x}) dx$ different from evaluating the integral in (a)? Do so.
 - (d) Evaluate each of the following integrals as well, keeping in mind the approach(es) used earlier in this problem:
 - $\int e^{2x} \sin(e^x) dx$
 - $\int e^{3x} \sin(e^{3x}) dx$
 - $\int xe^{x^2} \cos(e^{x^2}) \sin(e^{x^2}) dx$
3. For each of the following indefinite integrals, determine whether you would use u -substitution, integration by parts, neither*, or both to evaluate the integral. In each case, write one sentence to explain your reasoning, and include a statement of any substitutions used. (That is, if you decide in a problem to let $u = e^{3x}$, you should state that, as well as that $du = 3e^{3x} dx$.) Finally, use your chosen approach to evaluate each integral. (* one of the following problems does not have an elementary antiderivative and you are not expected to actually evaluate this integral; this will correspond with a choice of “neither” among those given.)
 - (a) $\int x^2 \cos(x^3) dx$
 - (b) $\int x^5 \cos(x^3) dx$ (Hint: $x^5 = x^2 \cdot x^3$)
 - (c) $\int x \ln(x^2) dx$
 - (d) $\int \sin(x^4) dx$
 - (e) $\int x^3 \sin(x^4) dx$
 - (f) $\int x^7 \sin(x^4) dx$

5.5 Other Options for Finding Algebraic Antiderivatives

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How does the method of partial fractions enable any rational function to be antidifferentiated?
- What role have integral tables historically played in the study of calculus and how can a table be used to evaluate integrals such as $\int \sqrt{a^2 + u^2} du$?
- What role can a computer algebra system play in the process of finding antiderivatives?

Introduction

In the preceding sections, we have learned two very specific antidifferentiation techniques: u -substitution and integration by parts. The former is used to reverse the chain rule, while the latter to reverse the product rule. But we have seen that each only works in very specialized circumstances. For example, while $\int xe^{x^2} dx$ may be evaluated by u -substitution and $\int xe^x dx$ by integration by parts, neither method provides a route to evaluate $\int e^{x^2} dx$. That fact is not a particular shortcoming of these two antidifferentiation techniques, as it turns out there does not exist an elementary algebraic antiderivative for e^{x^2} . Said differently, no matter what antidifferentiation methods we could develop and learn to execute, none of them will be able to provide us with a simple formula that does not involve integrals for a function $F(x)$ that satisfies $F'(x) = e^{x^2}$.

In this section of the text, our main goals are to better understand some classes of functions that can always be antidifferentiated, as well as to learn some options for so doing. At the same time, we want to recognize that there are many functions for which an algebraic formula for an antiderivative does not exist, and also appreciate the role that computing technology can play in helping us find antiderivatives of other complicated functions. Throughout, it is helpful to remember what we have learned so far: how to reverse the chain rule through u -substitution, how to reverse the product rule through integration by parts, and that overall, there are subtle and challenging issues to address when trying to find antiderivatives.

Preview Activity 5.5. For each of the indefinite integrals below, the main question is to decide whether the integral can be evaluated using u -substitution, integration by parts, a combination of the two, or neither. For integrals for which your answer is affirmative, state the substitution(s) you would use. It is not necessary to actually evaluate any of the integrals completely, unless the integral can be evaluated immediately using a familiar

basic antiderivative.

$$(a) \int x^2 \sin(x^3) dx, \int x^2 \sin(x) dx, \int \sin(x^3) dx, \int x^5 \sin(x^3) dx$$

$$(b) \int \frac{1}{1+x^2} dx, \int \frac{x}{1+x^2} dx, \int \frac{2x+3}{1+x^2} dx, \int \frac{e^x}{1+(e^x)^2} dx,$$

$$(c) \int x \ln(x) dx, \int \frac{\ln(x)}{x} dx, \int \ln(1+x^2) dx, \int x \ln(1+x^2) dx,$$

$$(d) \int x\sqrt{1-x^2} dx, \int \frac{1}{\sqrt{1-x^2}} dx, \int \frac{x}{\sqrt{1-x^2}} dx, \int \frac{1}{x\sqrt{1-x^2}} dx,$$

⌘

The Method of Partial Fractions

The method of partial fractions is used to integrate rational functions, and essentially involves reversing the process of finding a common denominator. For example, suppose we have the function $R(x) = \frac{5x}{x^2-x-2}$ and want to evaluate

$$\int \frac{5x}{x^2-x-2} dx.$$

Thinking algebraically, if we factor the denominator, we can see how R might come from the sum of two fractions of the form $\frac{A}{x-2} + \frac{B}{x+1}$. In particular, suppose that

$$\frac{5x}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}.$$

Multiplying both sides of this last equation by $(x-2)(x+1)$, we find that

$$5x = A(x+1) + B(x-2).$$

Since we want this equation to hold for every value of x , we can use insightful choices of specific x -values to help us find A and B . Taking $x = -1$, we have

$$5(-1) = A(0) + B(-3),$$

and thus $B = \frac{5}{3}$. Choosing $x = 2$, it follows

$$5(2) = A(3) + B(0),$$

so $A = \frac{10}{3}$. Therefore, we now know that

$$\int \frac{5x}{x^2-x-2} dx = \int \frac{10/3}{x-2} + \frac{5/3}{x+1} dx.$$

This equivalent integral expression is straightforward to evaluate, and hence we find that

$$\int \frac{5x}{x^2 - x - 2} dx = \frac{10}{3} \ln |x - 2| + \frac{5}{3} \ln |x + 1| + C.$$

It turns out that for any rational function $R(x) = \frac{P(x)}{Q(x)}$ where the degree of the polynomial P is less than⁷ the degree of the polynomial Q , the method of partial fractions can be used to rewrite the rational function as a sum of simpler rational functions of one of the following forms:

$$\frac{A}{x - c}, \quad \frac{A}{(x - c)^n}, \quad \text{or} \quad \frac{Ax + B}{x^2 + k}$$

where A , B , and c are real numbers, and k is a positive real number. Because each of these basic forms is one we can antidifferentiate, partial fractions enables us to antidifferentiate any rational function.

A computer algebra system such as *Maple*, *Mathematica*, or *WolframAlpha* can be used to find the partial fraction decomposition of any rational function. In *WolframAlpha*, entering

partial fraction 5x/(x^2-x-2)

results in the output

$$\frac{5x}{x^2 - x - 2} = \frac{10}{3(x - 2)} + \frac{5}{3(x + 1)}.$$

We will primarily use technology to generate partial fraction decompositions of rational functions, and then work from there to evaluate the integrals of interest using established methods.

Activity 5.13.

For each of the following problems, evaluate the integral by using the partial fraction decomposition provided.

(a) $\int \frac{1}{x^2 - 2x - 3} dx,$ given that $\frac{1}{x^2 - 2x - 3} = \frac{1/4}{x - 3} - \frac{1/4}{x + 1}$

(b) $\int \frac{x^2 + 1}{x^3 - x^2} dx,$ given that $\frac{x^2 + 1}{x^3 - x^2} = -\frac{1}{x} - \frac{1}{x^2} + \frac{2}{x - 1}$

(c) $\int \frac{x - 2}{x^4 + x^2} dx,$ given that $\frac{x - 2}{x^4 + x^2} = \frac{1}{x} - \frac{2}{x^2} + \frac{-x + 2}{1 + x^2}$

◀

⁷If the degree of P is greater than or equal to the degree of Q , long division may be used to write R as the sum of a polynomial plus a rational function where the numerator's degree is less than the denominator's.

Using an Integral Table

Calculus has a long history, with key ideas going back as far as Greek mathematicians in 400-300 BC. Its main foundations were first investigated and understood independently by Isaac Newton and Gottfried Wilhelm Leibniz in the late 1600s, making the modern ideas of calculus well over 300 years old. It is instructive to realize that until the late 1980s, the personal computer essentially did not exist, so calculus (and other mathematics) had to be done by hand for roughly 300 years. During the last 30 years, however, computers have revolutionized many aspects of the world we live in, including mathematics. In this section we take a short historical tour to precede the following discussion of the role computer algebra systems can play in evaluating indefinite integrals. In particular, we consider a class of integrals involving certain radical expressions that, until the advent of computer algebra systems, were often evaluated using an integral table.

As seen in the short table of integrals found in Appendix A, there are also many forms of integrals that involve $\sqrt{a^2 \pm w^2}$ and $\sqrt{w^2 - a^2}$. These integral rules can be developed using a technique known as *trigonometric substitution* that we choose to omit; instead, we will simply accept the results presented in the table. To see how these rules are needed and used, consider the differences among

$$\int \frac{1}{\sqrt{1-x^2}} dx, \quad \int \frac{x}{\sqrt{1-x^2}} dx, \quad \text{and} \quad \int \sqrt{1-x^2} dx.$$

The first integral is a familiar basic one, and results in $\arcsin(x) + C$. The second integral can be evaluated using a standard u -substitution with $u = 1 - x^2$. The third, however, is not familiar and does not lend itself to u -substitution.

In Appendix A, we find the rule

$$(8) \quad \int \sqrt{a^2 - u^2} du = \frac{u}{2} \sqrt{a^2 - u^2} + \frac{a^2}{2} \arcsin \frac{u}{a} + C.$$

Using the substitutions $a = 1$ and $u = x$ (so that $du = dx$), it follows that

$$\int \sqrt{1-x^2} dx = \frac{x}{2} \sqrt{1-x^2} - \frac{1}{2} \arcsin x + C.$$

One important point to note is that whenever we are applying a rule in the table, we are doing a u -substitution. This is especially key when the situation is more complicated than allowing $u = x$ as in the last example. For instance, say we wish to evaluate the integral

$$\int \sqrt{9 + 64x^2} dx.$$

Once again, we want to use Rule (3) from the table, but now do so with $a = 3$ and $u = 8x$; we also choose the “+” option in the rule. With this substitution, it follows that $du = 8dx$,

so $dx = \frac{1}{du}$. Applying this substitution,

$$\int \sqrt{9 + 64x^2} dx = \int \sqrt{9 + u^2} \cdot \frac{1}{8} du = \frac{1}{8} \int \sqrt{9 + u^2} du.$$

By Rule (3), we now find that

$$\begin{aligned} \int \sqrt{9 + 64x^2} dx &= \frac{1}{8} \left(\frac{u}{2} \sqrt{u^2 + 9} + \frac{9}{2} \ln |u + \sqrt{u^2 + 9}| + C \right) \\ &= \frac{1}{8} \left(\frac{8x}{2} \sqrt{64x^2 + 9} + \frac{9}{2} \ln |8x + \sqrt{64x^2 + 9}| + C \right). \end{aligned}$$

In problems such as this one, it is essential that we not forget to account for the factor of $\frac{1}{8}$ that must be present in the evaluation.

Activity 5.14.

For each of the following integrals, evaluate the integral using u -substitution and/or an entry from the table found in Appendix A.

(a) $\int \sqrt{x^2 + 4} dx$

(b) $\int \frac{x}{\sqrt{x^2 + 4}} dx$

(c) $\int \frac{2}{\sqrt{16 + 25x^2}} dx$

(d) $\int \frac{1}{x^2 \sqrt{49 - 36x^2}} dx$

◀

Using Computer Algebra Systems

A computer algebra system (CAS) is a computer program that is capable of executing symbolic mathematics. For a simple example, if we ask a CAS to solve the equation $ax^2 + bx + c = 0$ for the variable x , where a , b , and c are arbitrary constants, the program will return $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. While research to develop the first CAS dates to the 1960s, these programs became more common and publicly available in the early 1990s. Two prominent early examples are the programs *Maple* and *Mathematica*, which were among the first computer algebra systems to offer a graphical user interface. Today, *Maple* and *Mathematica* are exceptionally powerful professional software packages that are capable of executing an amazing array of sophisticated mathematical computations. They are also very expensive, as each is a proprietary program. The CAS *SAGE* is an open-source, free alternative to *Maple* and *Mathematica*.

For the purposes of this text, when we need to use a CAS, we are going to turn instead to a similar, but somewhat different computational tool, the web-based “computational knowledge engine” called *WolframAlpha*. There are two features of *WolframAlpha* that make it stand out from the CAS options mentioned above: (1) unlike *Maple* and *Mathematica*, *WolframAlpha* is free (provided we are willing to suffer through some pop-up advertising); and (2) unlike any of the three, the syntax in *WolframAlpha* is flexible. Think of *WolframAlpha* as being a little bit like doing a Google search: the program will interpret what is input, and then provide a summary of options.

If we want to have *WolframAlpha* evaluate an integral for us, we can provide it syntax such as

```
integrate x^2 dx
```

to which the program responds with

$$\int x^2 dx = \frac{x^3}{3} + \text{constant}.$$

While there is much to be enthusiastic about regarding CAS programs such as *WolframAlpha*, there are several things we should be cautious about: (1) a CAS only responds to exactly what is input; (2) a CAS can answer using powerful functions from highly advanced mathematics; and (3) there are problems that even a CAS cannot do without additional human insight.

Although (1) likely goes without saying, we have to be careful with our input: if we enter syntax that defines a function other than the problem of interest, the CAS will work with precisely the function we define. For example, if we are interested in evaluating the integral

$$\int \frac{1}{16 - 5x^2} dx,$$

and we mistakenly enter

```
integrate 1/16 - 5x^2 dx
```

a CAS will (correctly) reply with

$$\frac{1}{16}x - \frac{5}{3}x^3.$$

It is essential that we are sufficiently well-versed in antidifferentiation to recognize that this function cannot be the one that we seek: integrating a rational function such as $\frac{1}{16-5x^2}$, we expect the logarithm function to be present in the result.

Regarding (2), even for a relatively simple integral such as $\int \frac{1}{16-5x^2} dx$, some CASs will invoke advanced functions rather than simple ones. For instance, if we use *Maple* to execute the command

```
int(1/(16-5*x^2), x);
```

the program responds with

$$\int \frac{1}{16 - 5x^2} dx = \frac{\sqrt{5}}{20} \operatorname{arctanh}\left(\frac{\sqrt{5}}{4}x\right).$$

While this is correct (save for the missing arbitrary constant, which *Maple* never reports), the inverse hyperbolic tangent function is not a common nor familiar one; a simpler way to express this function can be found by using the partial fractions method, and happens to be the result reported by *WolframAlpha*:

$$\int \frac{1}{16 - 5x^2} dx = \frac{1}{8\sqrt{5}} \left(\log(4\sqrt{5} + 5\sqrt{x}) - \log(4\sqrt{5} - 5\sqrt{x}) \right) + \text{constant}.$$

Using sophisticated functions from more advanced mathematics is sometimes the way a CAS says to the user “I don’t know how to do this problem.” For example, if we want to evaluate

$$\int e^{-x^2} dx,$$

and we ask *WolframAlpha* to do so, the input

```
integrate exp(-x^2) dx
```

results in the output

$$\int e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \operatorname{erf}(x) + \text{constant}.$$

The function “ $\operatorname{erf}(x)$ ” is the *error function*, which is actually defined by an integral:

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$$

So, in producing output involving an integral, the CAS has basically reported back to us the very question we asked.

Finally, as remarked at (3) above, there are times that a CAS will actually fail without some additional human insight. If we consider the integral

$$\int (1+x)e^x \sqrt{1+x^2} e^{2x} dx$$

and ask *WolframAlpha* to evaluate

```
int (1+x) * exp(x) * sqrt(1+x^2 * exp(2x)) dx,
```

the program thinks for a moment and then reports

(no result found in terms of standard mathematical functions)

But in fact this integral is not that difficult to evaluate. If we let $u = xe^x$, then $du = (1+x)e^x dx$, which means that the preceding integral has form

$$\int (1+x)e^x \sqrt{1+x^2e^{2x}} dx = \int \sqrt{1+u^2} du,$$

which is a straightforward one for any CAS to evaluate.

So, the above observations regarding computer algebra systems lead us to proceed with some caution: while any CAS is capable of evaluating a wide range of integrals (both definite and indefinite), there are times when the result can mislead us. We must think carefully about the meaning of the output, whether it is consistent with what we expect, and whether or not it makes sense to proceed.

Summary

In this section, we encountered the following important ideas:

- The method of partial fractions enables any rational function to be antidifferentiated, because any polynomial function can be factored into a product of linear and irreducible quadratic terms. This allows any rational function to be written as the sum of a polynomial plus rational terms of the form $\frac{A}{(x-c)^n}$ (where n is a natural number) and $\frac{Bx+C}{x^2+k}$ (where k is a positive real number).
- Until the development of computing algebra systems, integral tables enabled students of calculus to more easily evaluate integrals such as $\int \sqrt{a^2+u^2} du$, where a is a positive real number. A short table of integrals may be found in Appendix A.
- Computer algebra systems can play an important role in finding antiderivatives, though we must be cautious to use correct input, to watch for unusual or unfamiliar advanced functions that the CAS may cite in its result, and to consider the possibility that a CAS may need further assistance or insight from us in order to answer a particular question.

Exercises

1. For each of the following integrals involving rational functions, (1) use a CAS to find the partial fraction decomposition of the integrand; (2) evaluate the integral of the resulting function without the assistance of technology; (3) use a CAS to evaluate the original integral to test and compare your result in (2).

(a) $\int \frac{x^3 + x + 1}{x^4 - 1} dx$

(b) $\int \frac{x^5 + x^2 + 3}{x^3 - 6x^2 + 11x - 6} dx$

$$(c) \int \frac{x^2 - x - 1}{(x - 3)^3} dx$$

2. For each of the following integrals involving radical functions, (1) use an appropriate u -substitution along with Appendix A to evaluate the integral without the assistance of technology, and (2) use a CAS to evaluate the original integral to test and compare your result in (1).

$$(a) \int \frac{1}{x\sqrt{9x^2 + 25}} dx$$

$$(b) \int x\sqrt{1 + x^4} dx$$

$$(c) \int e^x \sqrt{4 + e^{2x}} dx$$

$$(d) \int \frac{\tan(x)}{\sqrt{9 - \cos^2(x)}} dx$$

3. Consider the indefinite integral given by

$$\int \frac{\sqrt{x + \sqrt{1 + x^2}}}{x} dx.$$

- (a) Explain why u -substitution does not offer a way to simplify this integral by discussing at least two different options you might try for u .
- (b) Explain why integration by parts does not seem to be a reasonable way to proceed, either, by considering one option for u and dv .
- (c) Is there any line in the integral table in Appendix A that is helpful for this integral?
- (d) Evaluate the given integral using *WolframAlpha*. What do you observe?
-

5.6 Numerical Integration

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How do we accurately evaluate a definite integral such as $\int_0^1 e^{-x^2} dx$ when we cannot use the First Fundamental Theorem of Calculus because the integrand lacks an elementary algebraic antiderivative? Are there ways to generate accurate estimates without using extremely large values of n in Riemann sums?
- What is the Trapezoid Rule, and how is it related to left, right, and middle Riemann sums?
- How are the errors in the Trapezoid Rule and Midpoint Rule related, and how can they be used to develop an even more accurate rule?

Introduction

When we were first exploring the problem of finding the net-signed area bounded by a curve, we developed the concept of a Riemann sum as a helpful estimation tool and a key step in the definition of the definite integral. In particular, as we found in Section 4.2, recall that the left, right, and middle Riemann sums of a function f on an interval $[a, b]$ are denoted L_n , R_n , and M_n , with formulas

$$L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x = \sum_{i=0}^{n-1} f(x_i)\Delta x, \quad (5.11)$$

$$R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x = \sum_{i=1}^n f(x_i)\Delta x, \quad (5.12)$$

$$M_n = f(\bar{x}_1)\Delta x + f(\bar{x}_2)\Delta x + \cdots + f(\bar{x}_n)\Delta x = \sum_{i=1}^n f(\bar{x}_i)\Delta x, \quad (5.13)$$

where $x_0 = a$, $x_i = a + i\Delta x$, $x_n = b$, and $\Delta x = \frac{b-a}{n}$. For the middle sum, note that $\bar{x}_i = (x_{i-1} + x_i)/2$.

Further, recall that a Riemann sum is essentially a sum of (possibly signed) areas of rectangles, and that the value of n determines the number of rectangles, while our choice of left endpoints, right endpoints, or midpoints determines how we use the given function to find the heights of the respective rectangles we choose to use. Visually, we can see the similarities and differences among these three options in Figure 5.14, where we consider the function $f(x) = \frac{1}{20}(x-4)^3 + 7$ on the interval $[1, 8]$, and use 5 rectangles for each of

the Riemann sums.

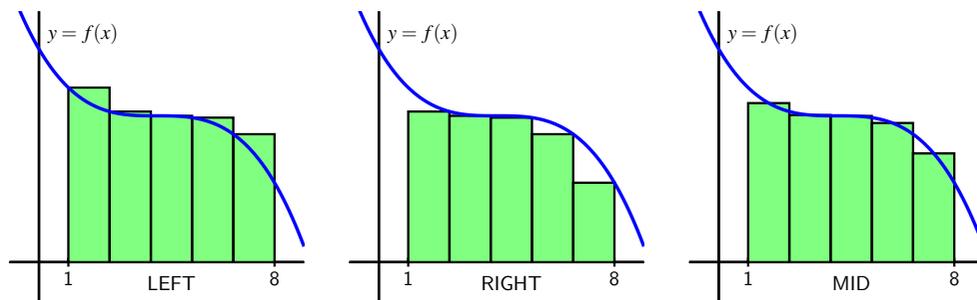


Figure 5.14: Left, right, and middle Riemann sums for $y = f(x)$ on $[1, 8]$ with 5 subintervals.

While it is a good exercise to compute a few Riemann sums by hand, just to ensure that we understand how they work and how varying the function, the number of subintervals, and the choice of endpoints or midpoints affects the result, it is of course the case that using computing technology is the best way to determine L_n , R_n , and M_n going forward. Any computer algebra system will offer this capability; as we saw in Preview Activity 4.3, a straightforward option that happens to also be freely available online is the applet⁸ at <http://gvsu.edu/s/a9>.

Note that we can adjust the formula for $f(x)$, the window of x - and y -values of interest, the number of subintervals, and the method. See Preview Activity 4.3 for any needed reminders on how the applet works.

In what follows in this section we explore several different alternatives, including left, right, and middle Riemann sums, for estimating definite integrals. One of our main goals in the upcoming section is to develop formulas that enable us to estimate definite integrals accurately without having to use exceptionally large numbers of rectangles.

Preview Activity 5.6. As we begin to investigate ways to approximate definite integrals, it will be insightful to compare results to integrals whose exact values we know. To that end, the following sequence of questions centers on $\int_0^3 x^2 dx$.

- Use the applet at <http://gvsu.edu/s/a9> with the function $f(x) = x^2$ on the window of x values from 0 to 3 to compute L_3 , the left Riemann sum with three subintervals.
- Likewise, use the applet to compute R_3 and M_3 , the right and middle Riemann sums with three subintervals, respectively.

⁸Marc Renault, Shippensburg University

- (c) Use the Fundamental Theorem of Calculus to compute the exact value of $I = \int_0^3 x^2 dx$.
- (d) We define the *error* in an approximation of a definite integral to be the difference between the integral's exact value and the approximation's value. What is the error that results from using L_3 ? From R_3 ? From M_3 ?
- (e) In what follows in this section, we will learn a new approach to estimating the value of a definite integral known as the Trapezoid Rule. The basic idea is to use trapezoids, rather than rectangles, to estimate the area under a curve. What is the formula for the area of a trapezoid with bases of length b_1 and b_2 and height h ?
- (f) Working by hand, estimate the area under $f(x) = x^2$ on $[0, 3]$ using three subintervals and three corresponding trapezoids. What is the error in this approximation? How does it compare to the errors you calculated in (d)?

✕

The Trapezoid Rule

Throughout our work to date with developing and estimating definite integrals, we have used the simplest possible quadrilaterals (that is, rectangles) to subdivide regions with complicated shapes. It is natural, however, to wonder if other familiar shapes might serve us even better. In particular, our goal is to be able to accurately estimate $\int_a^b f(x) dx$ without having to use extremely large values of n in Riemann sums.

To this end, we consider an alternative to L_n , R_n , and M_n , known as the *Trapezoid Rule*. The fundamental idea is simple: rather than using a rectangle to estimate the (signed) area bounded by $y = f(x)$ on a small interval, we use a trapezoid. For example, in Figure 5.15, we estimate the area under the pictured curve using three subintervals and the trapezoids that result from connecting the corresponding points on the curve with straight lines.

The biggest difference between the Trapezoid Rule and a left, right, or middle Riemann sum is that on each subinterval, the Trapezoid Rule uses two function values, rather than one, to estimate the (signed) area bounded by the curve. For instance, to compute D_1 , the area of the trapezoid generated by the curve $y = f(x)$ in Figure 5.15 on $[x_0, x_1]$, we observe that the left base of this trapezoid has length $f(x_0)$, while the right base has length $f(x_1)$. In addition, the height of this trapezoid is $x_1 - x_0 = \Delta x = \frac{b-a}{3}$. Since the area of a trapezoid is the average of the bases times the height, we have

$$D_1 = \frac{1}{2}(f(x_0) + f(x_1)) \cdot \Delta x.$$

Using similar computations for D_2 and D_3 , we find that T_3 , the trapezoidal approximation

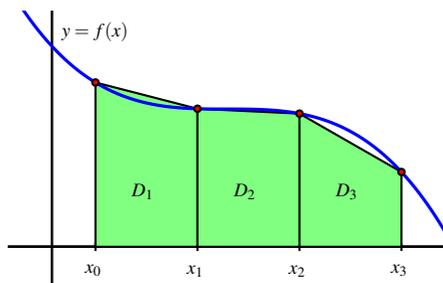


Figure 5.15: Estimating $\int_a^b f(x) dx$ using three subintervals and trapezoids, rather than rectangles, where $a = x_0$ and $b = x_3$.

to $\int_a^b f(x) dx$ is given by

$$\begin{aligned} T_3 &= D_1 + D_2 + D_3 \\ &= \frac{1}{2}(f(x_0) + f(x_1)) \cdot \Delta x + \frac{1}{2}(f(x_1) + f(x_2)) \cdot \Delta x + \frac{1}{2}(f(x_2) + f(x_3)) \cdot \Delta x. \end{aligned}$$

Because both left and right endpoints are being used, we recognize within the trapezoidal approximation the use of both left and right Riemann sums. In particular, rearranging the expression for T_3 by removing a factor of $\frac{1}{2}$, grouping the left endpoint evaluations of f , and grouping the right endpoint evaluations of f , we see that

$$T_3 = \frac{1}{2} [(f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x) + (f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x)]. \quad (5.14)$$

At this point, we observe that two familiar sums have arisen. Since the left Riemann sum L_3 is $L_3 = f(x_0)\Delta x + f(x_1)\Delta x + f(x_2)\Delta x$, and the right Riemann sum is $R_3 = f(x_1)\Delta x + f(x_2)\Delta x + f(x_3)\Delta x$, substituting L_3 and R_3 for the corresponding expressions in Equation 5.14, it follows that $T_3 = \frac{1}{2} [L_3 + R_3]$. We have thus seen the main ideas behind a very important result: using trapezoids to estimate the (signed) area bounded by a curve is the same as averaging the estimates generated by using left and right endpoints.

(The Trapezoid Rule) The trapezoidal approximation, T_n , of the definite integral $\int_a^b f(x) dx$ using n subintervals is given by the rule

$$\begin{aligned} T_n &= \frac{1}{2}(f(x_0) + f(x_1))\Delta x + \frac{1}{2}(f(x_1) + f(x_2))\Delta x + \cdots + \frac{1}{2}(f(x_{n-1}) + f(x_n))\Delta x. \\ &= \sum_{i=0}^{n-1} \frac{1}{2}(f(x_i) + f(x_{i+1}))\Delta x. \end{aligned}$$

Moreover, $T_n = \frac{1}{2} [L_n + R_n]$.

Activity 5.15.

In this activity, we explore the relationships among the errors generated by left, right, midpoint, and trapezoid approximations to the definite integral $\int_1^2 \frac{1}{x^2} dx$

- Use the First FTC to evaluate $\int_1^2 \frac{1}{x^2} dx$ exactly.
- Use appropriate computing technology to compute the following approximations for $\int_1^2 \frac{1}{x^2} dx$: T_4 , M_4 , T_8 , and M_8 .
- Let the *error* of an approximation be the difference between the exact value of the definite integral and the resulting approximation. For instance, if we let $E_{T,4}$ represent the error that results from using the trapezoid rule with 4 subintervals to estimate the integral, we have

$$E_{T,4} = \int_1^2 \frac{1}{x^2} dx - T_4.$$

Similarly, we compute the error of the midpoint rule approximation with 8 subintervals by the formula

$$E_{M,8} = \int_1^2 \frac{1}{x^2} dx - M_8.$$

Based on your work in (a) and (b) above, compute $E_{T,4}$, $E_{T,8}$, $E_{M,4}$, $E_{M,8}$.

- Which rule consistently over-estimates the exact value of the definite integral? Which rule consistently under-estimates the definite integral?
- What behavior(s) of the function $f(x) = \frac{1}{x^2}$ lead to your observations in (d)?

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Comparing the Midpoint and Trapezoid Rules

We know from the definition of the definite integral of a continuous function f , that if we let n be large enough, we can make the value of any of the approximations L_n , R_n , and M_n as close as we'd like (in theory) to the exact value of $\int_a^b f(x) dx$. Thus, it may be natural to wonder why we ever use any rule other than L_n or R_n (with a sufficiently large n value) to estimate a definite integral. One of the primary reasons is that as $n \rightarrow \infty$, $\Delta x = \frac{b-a}{n} \rightarrow 0$, and thus in a Riemann sum calculation with a large n value, we end up multiplying by a number that is very close to zero. Doing so often generates roundoff error, as representing numbers close to zero accurately is a persistent challenge for computers.

Hence, we are exploring ways by which we can estimate definite integrals to high levels of precision, but without having to use extremely large values of n . Paying close attention

to patterns in errors, such as those observed in Activity 5.15, is one way to begin to see some alternate approaches.

To begin, we make a comparison of the errors in the Midpoint and Trapezoid rules from two different perspectives. First, consider a function of consistent concavity on a given interval, and picture approximating the area bounded on that interval by both the Midpoint and Trapezoid rules using a single subinterval. As seen in Figure 5.16, it

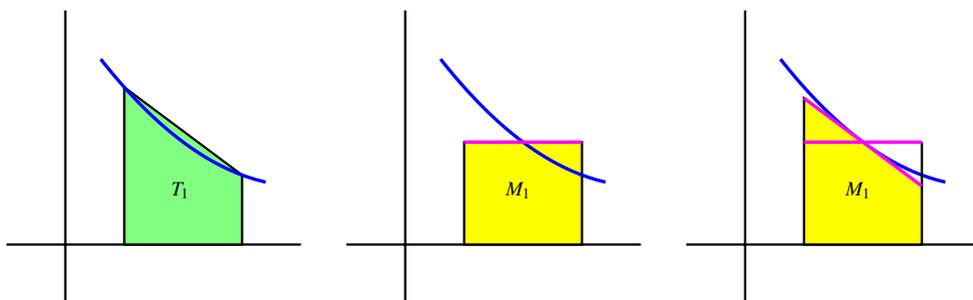


Figure 5.16: Estimating $\int_a^b f(x) dx$ using a single subinterval: at left, the trapezoid rule; in the middle, the midpoint rule; at right, a modified way to think about the midpoint rule.

is evident that whenever the function is concave up on an interval, the Trapezoid Rule with one subinterval, T_1 , will overestimate the exact value of the definite integral on that interval. Moreover, from a careful analysis of the line that bounds the top of the rectangle for the Midpoint Rule (shown in magenta), we see that if we rotate this line segment until it is tangent to the curve at the point on the curve used in the Midpoint Rule (as shown at right in Figure 5.16), the resulting trapezoid has the same area as M_1 , and this value is less than the exact value of the definite integral. Hence, when the function is concave up on the interval, M_1 underestimates the integral's true value.

These observations extend easily to the situation where the function's concavity remains consistent but we use higher values of n in the Midpoint and Trapezoid Rules. Hence, whenever f is concave up on $[a, b]$, T_n will overestimate the value of $\int_a^b f(x) dx$, while M_n will underestimate $\int_a^b f(x) dx$. The reverse observations are true in the situation where f is concave down.

Next, we compare the size of the errors between M_n and T_n . Again, we focus on M_1 and T_1 on an interval where the concavity of f is consistent. In Figure 5.17, where the error of the Trapezoid Rule is shaded in red, while the error of the Midpoint Rule is shaded lighter red, it is visually apparent that the error in the Trapezoid Rule is more significant. To see how much more significant, let's consider two examples and some particular computations.

If we let $f(x) = 1 - x^2$ and consider $\int_0^1 f(x) dx$, we know by the First FTC that the

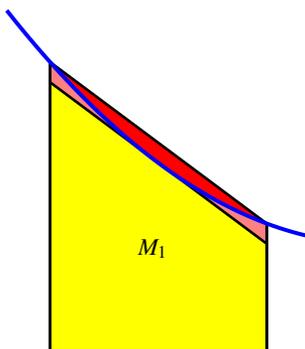


Figure 5.17: Comparing the error in estimating $\int_a^b f(x) dx$ using a single subinterval: in red, the error from the Trapezoid rule; in light red, the error from the Midpoint rule.

exact value of the integral is

$$\int_0^1 (1 - x^2) dx = x - \frac{x^3}{3} \Big|_0^1 = \frac{2}{3}.$$

Using appropriate technology to compute M_4 , M_8 , T_4 , and T_8 , as well as the corresponding errors $E_{M,4}$, $E_{M,8}$, $E_{T,4}$, and $E_{T,8}$, as we did in Activity 5.15, we find the results summarized in Table 5.1. Note that in the table, we also include the approximations and their errors for the example $\int_1^2 \frac{1}{x^2} dx$ from Activity 5.15.

	$\int_0^1 (1 - x^2) dx = 0.\bar{6}$	error	$\int_1^2 \frac{1}{x^2} dx = 0.5$	error
T_4	0.65625	-0.0104166667	0.5089937642	0.0089937642
M_4	0.671875	0.0052083333	0.4955479365	-0.0044520635
T_8	0.6640625	-0.0026041667	0.5022708502	0.0022708502
M_8	0.66796875	0.0013020833	0.4988674899	-0.0011325101

Table 5.1: Calculations of T_4 , M_4 , T_8 , and M_8 , along with corresponding errors, for the definite integrals $\int_0^1 (1 - x^2) dx$ and $\int_1^2 \frac{1}{x^2} dx$.

Recall that for a given function f and interval $[a, b]$, $E_{T,4} = \int_a^b f(x) dx - T_4$ calculates the difference between the exact value of the definite integral and the approximation generated by the Trapezoid Rule with $n = 4$. If we look at not only $E_{T,4}$, but also the other errors generated by using T_n and M_n with $n = 4$ and $n = 8$ in the two examples noted in Table 5.1, we see an evident pattern. Not only is the sign of the error (which

measures whether the rule generates an over- or under-estimate) tied to the rule used and the function's concavity, but the magnitude of the errors generated by T_n and M_n seems closely connected. In particular, the errors generated by the Midpoint Rule seem to be about half the size of those generated by the Trapezoid Rule.

That is, we can observe in both examples that $E_{M,4} \approx -\frac{1}{2}E_{T,4}$ and $E_{M,8} \approx -\frac{1}{2}E_{T,8}$, which demonstrates a property of the Midpoint and Trapezoid Rules that turns out to hold in general: for a function of consistent concavity, the error in the Midpoint Rule has the opposite sign and approximately half the magnitude of the error of the Trapezoid Rule. Said symbolically,

$$E_{M,n} \approx -\frac{1}{2}E_{T,n}.$$

This important relationship suggests a way to combine the Midpoint and Trapezoid Rules to create an even more accurate approximation to a definite integral.

Simpson's Rule

When we first developed the Trapezoid Rule, we observed that it can equivalently be viewed as resulting from the average of the Left and Right Riemann sums:

$$T_n = \frac{1}{2}(L_n + R_n).$$

Whenever a function is always increasing or always decreasing on the interval $[a, b]$, one of L_n and R_n will over-estimate the true value of $\int_a^b f(x) dx$, while the other will under-estimate the integral. Said differently, the errors found in L_n and R_n will have opposite signs; thus, averaging L_n and R_n eliminates a considerable amount of the error present in the respective approximations. In a similar way, it makes sense to think about averaging M_n and T_n in order to generate a still more accurate approximation.

At the same time, we've just observed that M_n is typically about twice as accurate as T_n . Thus, we instead choose to use the weighted average

$$S_{2n} = \frac{2M_n + T_n}{3}. \quad (5.15)$$

The rule for S_{2n} giving by Equation 5.15 is usually known as *Simpson's Rule*.⁹ Note that we use " S_{2n} " rather than " S_n " since the n points the Midpoint Rule uses are different from the n points the Trapezoid Rule uses, and thus Simpson's Rule is using $2n$ points at which to evaluate the function. We build upon the results in Table 5.1 to see the approximations generated by Simpson's Rule. In particular, in Table 5.2, we include all of the results in

⁹Thomas Simpson was an 18th century mathematician; his idea was to extend the Trapezoid rule, but rather than using straight lines to build trapezoids, to use quadratic functions to build regions whose area was bounded by parabolas (whose areas he could find exactly). Simpson's Rule is often developed from the more sophisticated perspective of using interpolation by quadratic functions.

Table 5.1, but include additional results for $S_8 = \frac{2M_4+T_4}{3}$ and $S_{16} = \frac{2M_8+T_8}{3}$.

	$\int_0^1 (1-x^2) dx = 0.\bar{6}$	error	$\int_1^2 \frac{1}{x^2} dx = 0.5$	error
T_4	0.65625	-0.0104166667	0.5089937642	0.0089937642
M_4	0.671875	0.0052083333	0.4955479365	-0.0044520635
S_8	0.6666666667	0	0.5000298792	0.0000298792
T_8	0.6640625	-0.0026041667	0.5022708502	0.0022708502
M_8	0.66796875	0.0013020833	0.4988674899	-0.0011325101
S_{16}	0.6666666667	0	0.5000019434	0.0000019434

Table 5.2: Table 5.1 updated to include S_8 , S_{16} , and the corresponding errors.

The results seen in Table 5.2 are striking. If we consider the S_{16} approximation of $\int_1^2 \frac{1}{x^2} dx$, the error is only $E_{S,16} = 0.0000019434$. By contrast, $L_8 = 0.5491458502$, so the error of that estimate is $E_{L,8} = -0.0491458502$. Moreover, we observe that generating the approximations for Simpson's Rule is almost no additional work: once we have L_n , R_n , and M_n for a given value of n , it is a simple exercise to generate T_n , and from there to calculate S_{2n} . Finally, note that the error in the Simpson's Rule approximations of $\int_0^1 (1-x^2) dx$ is zero!¹⁰

These rules are not only useful for approximating definite integrals such as $\int_0^1 e^{-x^2} dx$, for which we cannot find an elementary antiderivative of e^{-x^2} , but also for approximating definite integrals in the setting where we are given a function through a table of data.

Activity 5.16.

A car traveling along a straight road is braking and its velocity is measured at several different points in time, as given in the following table. Assume that v is continuous, always decreasing, and always decreasing at a decreasing rate, as is suggested by the data.

seconds, t	0	0.3	0.6	0.9	1.2	1.5	1.8
Velocity in ft/sec, $v(t)$	100	99	96	90	80	50	0

- Plot the given data on the set of axes provided in Figure 5.18 with time on the horizontal axis and the velocity on the vertical axis.
- What definite integral will give you the exact distance the car traveled on $[0, 1.8]$?

¹⁰Similar to how the Midpoint and Trapezoid approximations are exact for linear functions, Simpson's Rule approximations are exact for quadratic and cubic functions. See additional discussion on this issue later in the section and in the exercises.

- (c) Estimate the total distance traveled on $[0, 1.8]$ by computing L_3 , R_3 , and T_3 . Which of these under-estimates the true distance traveled?
- (d) Estimate the total distance traveled on $[0, 1.8]$ by computing M_3 . Is this an over- or under-estimate? Why?
- (e) Using your results from (c) and (d), improve your estimate further by using Simpson's Rule.
- (f) What is your best estimate of the average velocity of the car on $[0, 1.8]$? Why? What are the units on this quantity?

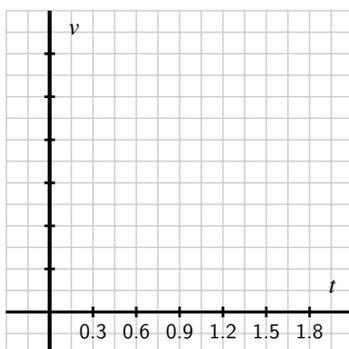


Figure 5.18: Axes for plotting the data in Activity 5.16.

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Overall observations regarding L_n , R_n , T_n , M_n , and S_{2n} .

As we conclude our discussion of numerical approximation of definite integrals, it is important to summarize general trends in how the various rules over- or under-estimate the true value of a definite integral, and by how much. To revisit some past observations and see some new ones, we consider the following activity.

Activity 5.17.

Consider the functions $f(x) = 2 - x^2$, $g(x) = 2 - x^3$, and $h(x) = 2 - x^4$, all on the interval $[0, 1]$. For each of the questions that require a numerical answer in what follows, write your answer exactly in fraction form.

- (a) On the three sets of axes provided in Figure 5.19, sketch a graph of each function on the interval $[0, 1]$, and compute L_1 and R_1 for each. What do you observe?

- (b) Compute M_1 for each function to approximate $\int_0^1 f(x) dx$, $\int_0^1 g(x) dx$, and $\int_0^1 h(x) dx$, respectively.
- (c) Compute T_1 for each of the three functions, and hence compute S_1 for each of the three functions.
- (d) Evaluate each of the integrals $\int_0^1 f(x) dx$, $\int_0^1 g(x) dx$, and $\int_0^1 h(x) dx$ exactly using the First FTC.
- (e) For each of the three functions f , g , and h , compare the results of L_1 , R_1 , M_1 , T_1 , and S_2 to the true value of the corresponding definite integral. What patterns do you observe?

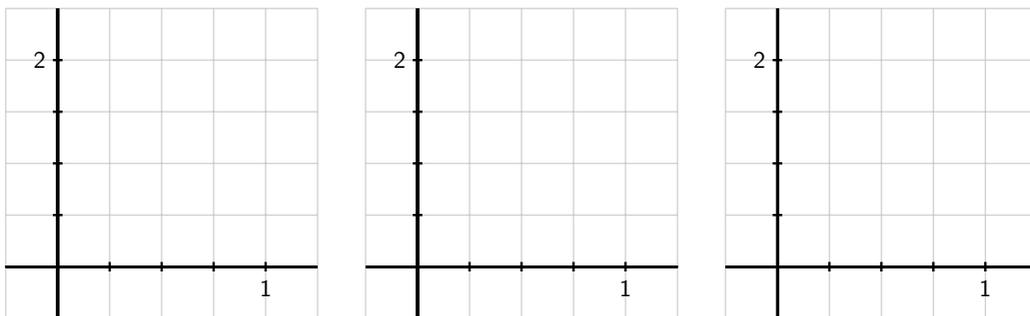


Figure 5.19: Axes for plotting the functions in Activity 5.17.

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The results seen in the examples in Activity 5.17 generalize nicely. For instance, for any function f that is decreasing on $[a, b]$, L_n will over-estimate the exact value of $\int_a^b f(x) dx$, and for any function f that is concave down on $[a, b]$, M_n will over-estimate the exact value of the integral. An excellent exercise is to write a collection of scenarios of possible function behavior, and then categorize whether each of L_n , R_n , T_n , and M_n is an over- or under-estimate.

Finally, we make two important notes about Simpson's Rule. When T. Simpson first developed this rule, his idea was to replace the function f on a given interval with a quadratic function that shared three values with the function f . In so doing, he guaranteed that this new approximation rule would be exact for the definite integral of any quadratic polynomial. In one of the pleasant surprises of numerical analysis, it turns out that even though it was designed to be exact for quadratic polynomials, Simpson's Rule is exact for any cubic polynomial: that is, if we are interested in an integral such as $\int_2^5 (5x^3 - 2x^2 + 7x - 4) dx$, S_{2n} will always be exact, regardless of the value of n . This is just one more piece of evidence that shows how effective Simpson's Rule is as an approximation

tool for estimating definite integrals.¹¹

Summary

In this section, we encountered the following important ideas:

- For a definite integral such as $\int_0^1 e^{-x^2} dx$ when we cannot use the First Fundamental Theorem of Calculus because the integrand lacks an elementary algebraic antiderivative, we can estimate the integral's value by using a sequence of Riemann sum approximations. Typically, we start by computing L_n , R_n , and M_n for one or more chosen values of n .
 - The Trapezoid Rule, which estimates $\int_a^b f(x) dx$ by using trapezoids, rather than rectangles, can also be viewed as the average of Left and Right Riemann sums. That is, $T_n = \frac{1}{2}(L_n + R_n)$.
 - The Midpoint Rule is typically twice as accurate as the Trapezoid Rule, and the signs of the respective errors of these rules are opposites. Hence, by taking the weighted average $S_n = \frac{2M_n + T_n}{3}$, we can build a much more accurate approximation to $\int_a^b f(x) dx$ by using approximations we have already computed. The rule for S_n is known as Simpson's Rule, which can also be developed by approximating a given continuous function with pieces of quadratic polynomials.
-

Exercises

1. Consider the definite integral $\int_0^1 x \tan(x) dx$.
 - (a) Explain why this integral cannot be evaluated exactly by using either u -substitution or by integrating by parts.
 - (b) Using 4 subintervals, compute L_4 , R_4 , M_4 , T_4 , and S_4 .
 - (c) Which of the approximations in (b) is an over-estimate to the true value of $\int_0^1 x \tan(x) dx$? Which is an under-estimate? How do you know?
2. For an unknown function $f(x)$, the following information is known.
 - f is continuous on $[3, 6]$;
 - f is either always increasing or always decreasing on $[3, 6]$;
 - f has the same concavity throughout the interval $[3, 6]$;
 - As approximations to $\int_3^6 f(x) dx$, $L_4 = 7.23$, $R_4 = 6.75$, and $M_4 = 7.05$.

¹¹One reason that Simpson's Rule is so effective is that S_{2n} benefits from using $2n + 1$ points of data. Because it combines M_n , which uses n midpoints, and T_n , which uses the $n + 1$ endpoints of the chosen subintervals, S_{2n} takes advantage of the maximum amount of information we have when we know function values at the endpoints and midpoints of n subintervals.

- (a) Is f increasing or decreasing on $[3, 6]$? What data tells you?
- (b) Is f concave up or concave down on $[3, 6]$? Why?
- (c) Determine the best possible estimate you can for $\int_3^6 f(x) dx$, based on the given information.
3. The rate at which water flows through Table Rock Dam on the White River in Branson, MO, is measured in thousands of cubic feet per second (TCFS). As engineers open the floodgates, flow rates are recorded according to the following chart.

seconds, t	0	10	20	30	40	50	60
flow in TCFS, $r(t)$	2000	2100	2400	3000	3900	5100	6500

- (a) What definite integral measures the total volume of water to flow through the dam in the 60 second time period provided by the table above?
- (b) Use the given data to calculate M_n for the largest possible value of n to approximate the integral you stated in (a). Do you think M_n over- or under-estimates the exact value of the integral? Why?
- (c) Approximate the integral stated in (a) by calculating S_n for the largest possible value of n , based on the given data.
- (d) Compute $\frac{1}{60}S_n$ and $\frac{2000+2100+2400+3000+3900+5100+6500}{7}$. What quantity do both of these values estimate? Which is a more accurate approximation?
-

