Chapter 7

Differential Equations

7.1 An Introduction to Differential Equations

> **Motivating Questions**
>
> *In this section, we strive to understand the ideas generated by the following important questions:*
>
> - What is a differential equation and what kinds of information can it tell us?
>
> - How do differential equations arise in the world around us?
>
> - What do we mean by a solution to a differential equation?

Introduction

In previous chapters, we have seen that a function's derivative tells us the rate at which the function is changing. More recently, the Fundamental Theorem of Calculus helped us to determine the total change of a function over an interval when we know the function's rate of change. For instance, an object's velocity tells us the rate of change of that object's position. By integrating the velocity over a time interval, we may determine by how much the position changes over that time interval. In particular, if we know where the object is at the beginning of that interval, then we have enough information to accurately predict where it will be at the end of the interval.

In this chapter, we will introduce the concept of *differential equations* and explore this idea in more depth. Simply said, a differential equation is an equation that provides a description of a function's derivative, which means that it tells us the function's rate of change. Using this information, we would like to learn as much as possible about the function itself. For instance, we would ideally like to have an algebraic description of the

function. As we'll see, this may be too much to ask in some situations, but we will still be able to make accurate approximations.

Preview Activity 7.1. The position of a moving object is given by the function $s(t)$, where s is measured in feet and t in seconds. We determine that the velocity is $v(t) = 4t + 1$ feet per second.

(a) How much does the position change over the time interval $[0, 4]$?

(b) Does this give you enough information to determine $s(4)$, the position at time $t = 4$? If so, what is $s(4)$? If not, what additional information would you need to know to determine $s(4)$?

(c) Suppose you are told that the object's initial position $s(0) = 7$. Determine $s(2)$, the object's position 2 seconds later.

(d) If you are told instead that the object's initial position is $s(0) = 3$, what is $s(2)$?

(e) If we only know the velocity $v(t) = 4t + 1$, is it possible that the object's position at all times is $s(t) = 2t^2 + t - 4$? Explain how you know.

(f) Are there other possibilities for $s(t)$? If so, what are they?

(g) If, in addition to knowing the velocity function is $v(t) = 4t + 1$, we know the initial position $s(0)$, how many possibilities are there for $s(t)$?

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What is a differential equation?

A differential equation is an equation that describes the derivative, or derivatives, of a function that is unknown to us. For instance, the equation

$$\frac{dy}{dx} = x \sin x$$

is a differential equation since it describes the derivative of a function $y(x)$ that is unknown to us.

   As many important examples of differential equations involve quantities that change in time, the independent variable in our discussion will frequently be time t. For instance, in the preview activity, we considered the differential equation

$$\frac{ds}{dt} = 4t + 1.$$

Knowing the velocity and the starting position of the object, we were able to find the position at any later time.

Because differential equations describe the derivative of a function, they give us information about how that function changes. Our goal will be to take this information and use it to predict the value of the function in the future; in this way, differential equations provide us with something like a crystal ball.

Differential equations arise frequently in our every day world. For instance, you may hear a bank advertising:

Your money will grow at a 3% annual interest rate with us.

This innocuous statement is really a differential equation. Let's translate: $A(t)$ will be amount of money you have in your account at time t. On one hand, the rate at which your money grows is the derivative dA/dt. On the other hand, we are told that this rate is $0.03A$. This leads to the differential equation

$$\frac{dA}{dt} = 0.03A.$$

This differential equation has a slightly different feel than the previous equation $\frac{ds}{dt} = 4t + 1$. In the earlier example, the rate of change depends only on the independent variable t, and we may find $s(t)$ by integrating the velocity $4t + 1$. In the banking example, however, the rate of change depends on the dependent variable A, so we'll need some new techniques in order to find $A(t)$.

Activity 7.1.

Express the following statements as differential equations. In each case, you will need to introduce notation to describe the important quantities in the statement so be sure to clearly state what your notation means.

(a) The population of a town grows continuously at an annual rate of 1.25%.

(b) A radioactive sample loses 5.6% of its mass every day.

(c) You have a bank account that continuously earns 4% interest every year. At the same time, you withdraw money continually from the account at the rate of $1000 per year.

(d) A cup of hot chocolate is sitting in a 70° room. The temperature of the hot chocolate cools continuously by 10% of the difference between the hot chocolate's temperature and the room temperature every minute.

(e) A can of cold soda is sitting in a 70° room. The temperature of the soda warms continuously at the rate of 10% of the difference between the soda's temperature and the room's temperature every minute.

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Differential equations in the world around us

As we have noted, differential equations give a natural way to describe phenomena we see in the real world. For instance, physical principles are frequently expressed as a description of how a quantity changes. A good example is Newton's Second Law, an important physcial principle that says:

The product of an object's mass and acceleration equals the force applied to it.

For instance, when gravity acts on an object near the earth's surface, it exerts a force equal to mg, the mass of the object times the gravitational constant g. We therefore have

$$ma = mg, \text{ or}$$
$$\frac{dv}{dt} = g,$$

where v is the velocity of the object, and $g = 9.8$ meters per second squared. Notice that this physical principle does not tell us what the object's velocity is, but rather how the object's velocity changes.

Activity 7.2.

Shown below are two graphs depicting the velocity of falling objects. On the left is the velocity of a skydiver, while on the right is the velocity of a meteorite entering the Earth's atmosphere.
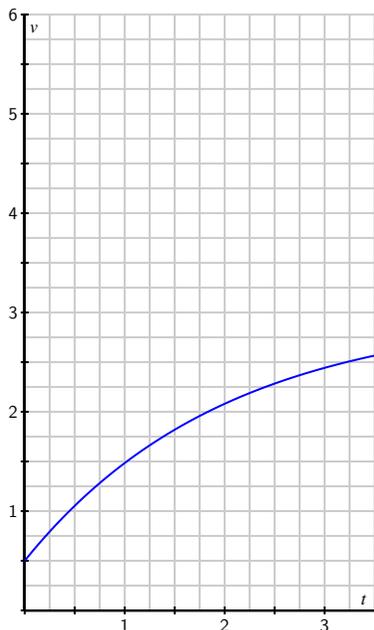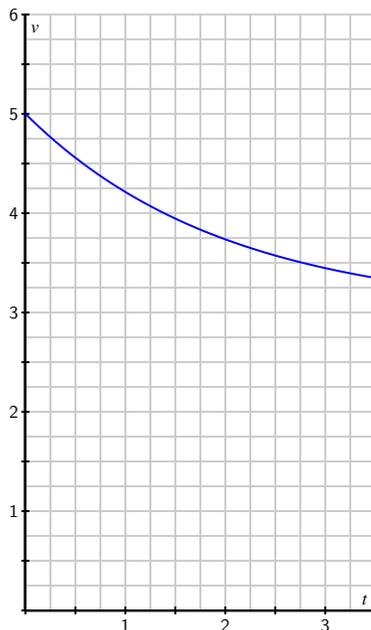
Skydiver's velocity　　　　　Meteorite's velocity

(a) Begin with the skydiver's velocity and use the given graph to measure the rate of change dv/dt when the velocity is $v = 0.5, 1.0, 1.5, 2.0$, and 2.5. Plot your values on the graph below. You will want to think carefully about this: you are plotting the derivative dv/dt as a function of *velocity.*

(b) Now do the same thing with the meteorite's velocity: use the given graph to measure the rate of change dv/dt when the velocity is $v = 3.5, 4.0, 4.5$, and 5.0. Plot your values on the graph above.

(c) You should find that all your points lie on a line. Write the equation of this line being careful to use proper notation for the quantities on the horizontal and vertical axes.

(d) The relationship you just found is a differential equation. Write a complete sentence that explains its meaning.

(e) By looking at the differential equation, determine the values of the velocity for which the velocity increases.

(f) By looking at the differential equation, determine the values of the velocity for which the velocity decreases.

(g) By looking at the differential equation, determine the values of the velocity for which the velocity remains constant.

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The point of this activity is to demonstrate how differential equations model processes in the real world. In this example, two factors are influencing the velocities: gravity and wind resistance. The differential equation describes how these factors influence the rate of change of the objects' velocities.

Solving a differential equation

We have said that a differential equation is an equation that describes the derivative, or derivatives, of a function that is unknown to us. By a *solution* to a differential equation, we mean simply a function that satisies this description.

For instance, the first differential equation we looked at is

$$\frac{ds}{dt} = 4t + 1,$$

which describes an unknown function $s(t)$. We may check that $s(t) = 2t^2 + t$ is a solution because it satisfies this description. Notice that $s(t) = 2t^2 + t + 4$ is also a solution.

If we have a candidate for a solution, it is straightforward to check whether it is a solution or not. Before we demonstrate, however, let's consider the same issue in a simpler context. Suppose we are given the equation $2x^2 - 2x = 2x + 6$ and asked whether $x = 3$ is a solution. To answer this question, we could rewrite the variable x in the equation with the symbol $\square$:

$$2\square^2 - 2\square = 2\square + 6.$$

To determine whether $x = 3$ is a solution, we can investigate the value of each side of the equation separately when the value 3 is placed in $\square$ and see if indeed the two resulting values are equal. Doing so, we observe that

$$2\square^2 - 2\square = 2 \cdot 3^2 - 2 \cdot 3 = 12,$$

and

$$2\square + 6 = 2 \cdot 3 + 6 = 12.$$

Therefore, $x = 3$ is indeed a solution.

We will do the same thing with differential equations. Consider the differential equation

$$\frac{dv}{dt} = 1.5 - 0.5v, \text{ or}$$
$$\frac{d\square}{dt} = 1.5 - 0.5\square.$$

Let's ask whether $v(t) = 3 - 2e^{-0.5t}$ is a solution[1]. Using this formula for v, observe first that

$$\frac{dv}{dt} = \frac{d\square}{dt} = \frac{d}{dt}[3 - 2e^{-0.5t}] = -2e^{-0.5t} \cdot (-0.5) = e^{-0.5t}$$

and

$$1.5 - 0.5v = 1.5 - 0.5\square = 1.5 - 0.5(3 - 2e^{-0.5t}) = 1.5 - 1.5 + e^{-0.5t} = e^{-0.5t}.$$

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[1]At this time, don't worry about why we chose this function; we will learn techniques for finding solutions to differential equations soon enough.

Since $\frac{dv}{dt}$ and $1.5 - 0.5v$ agree for all values of t when $v = 3 - 2e^{-0.5t}$, we have indeed found a solution to the differential equation.

Activity 7.3.

Consider the differential equation

$$\frac{dv}{dt} = 1.5 - 0.5v.$$

Which of the following functions are solutions of this differential equation?

(a) $v(t) = 1.5t - 0.25t^2$.

(b) $v(t) = 3 + 2e^{-0.5t}$.

(c) $v(t) = 3$.

(d) $v(t) = 3 + Ce^{-0.5t}$ where C is any constant.

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This activity shows us something interesting. Notice that the differential equation has infinitely many solutions, which are parametrized by the constant C in $v(t) = 3 + Ce^{-0.5t}$. In Figure 7.1, we see the graphs of these solutions for a few values of C, as labeled.
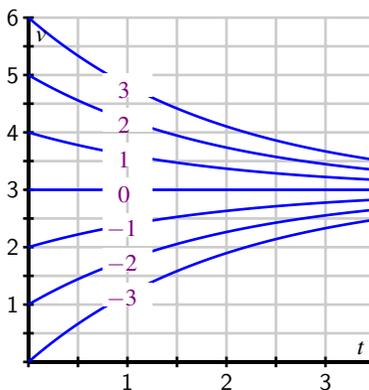

Figure 7.1: The family of solutions to the differential equation $\frac{dv}{dt} = 1.5 - 0.5v$.

Notice that the value of C is connected to the initial value of the velocity $v(0)$, since $v(0) = 3 + C$. In other words, while the differential equation describes how the velocity changes as a function of the velocity itself, this is not enough information to determine the velocity uniquely: we also need to know the initial velocity. For this reason, differential equations will typically have infinitely many solutions, one corresponding to each initial value. We have seen this phenomenon before, such as when given the velocity of a moving object $v(t)$, we were not able to uniquely determine the object's position unless we also know its initial position.

If we are given a differential equation and an initial value for the unknown function, we say that we have an *initial value problem.* For instance,

$$\frac{dv}{dt} = 1.5 - 0.5v, \ v(0) = 0.5$$

is an initial value problem. In this situation, we know the value of v at one time and we know how v is changing. Consequently, there should be exactly one function v that satisfies the initial value problem.

This demonstrates the following important general property of initial value problems.

> Initial value problems that are "well behaved" have exactly one solution, which exists in some interval around the initial point.

We won't worry about what "well behaved" means—it is a technical condition that will be satisfied by all the differential equations we consider.

To close this section, we note that differential equations may be classified based on certain characteristics they may possess. Indeed, you may see many different types of differential equations in a later course in differential equations. For now, we would like to introduce a few terms that are used to describe differential equations.

A *first-order* differential equation is one in which only the first derivative of the function occurs. For this reason,

$$\frac{dv}{dt} = 1.5 - 0.5v$$

is a first-order equation while

$$\frac{d^2y}{dt^2} = -10y$$

is a second-order equation.

A differential equation is *autonomous* if the independent variable does not appear in the description of the derivative. For instance,

$$\frac{dv}{dt} = 1.5 - 0.5v$$

is autonomous because the description of the derivative dv/dt does not depend on time. The equation

$$\frac{dy}{dt} = 1.5t - 0.5y,$$

however, is not autonomous.

Summary

*In this section, we encountered the following important ideas:*

- A differential equation is simply an equation that describes the derivative(s) of an unknown function.

- Physical principles, as well as some everyday situations, often describe how a quantity changes, which lead to differential equations.

- A solution to a differential equation is a function whose derivatives satisfy the equation's description. Differential equations typically have infinitely many solutions, parametrized by the initial values.

Exercises

1. Suppose that $T(t)$ represents the temperature of a cup of coffee set out in a room, where T is expressed in degrees Fahrenheit and t in minutes. A physical principle known as Newton's Law of Cooling tells us that

$$\frac{dT}{dt} = -\frac{1}{15}T + 5.$$

(a) Supposes that $T(0) = 105$. What does the differential equation give us for the value of $\frac{dT}{dt}|_{T=105}$? Explain in a complete sentence the meaning of these two facts.

(b) Is T increasing or decreasing at $t = 0$?

(c) What is the approximate temperature at $t = 1$?

(d) On the graph below, make a plot of dT/dt as a function of T.
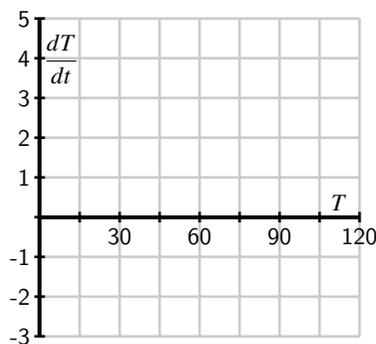
(e) For which values of T does T increase? For which values of T does T decrease?

(f) What do you think is the temperature of the room? Explain your thinking.

(g) Verify that $T(t) = 75 + 30e^{-t/15}$ is the solution to the differential equation with initial value $T(0) = 105$. What happens to this solution after a long time?

2. Suppose that the population of a particular species is described by the function $P(t)$, where P is expressed in millions. Suppose further that the population's rate of change is governed by the differential equation

$$\frac{dP}{dt} = f(P)$$

where $f(P)$ is the function graphed below.
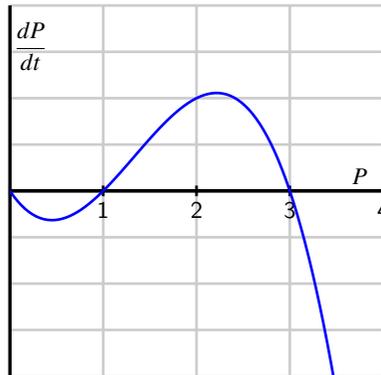
(a) For which values of the population P does the population increase?

(b) For which values of the population P does the population decrease?

(c) If $P(0) = 3$, how will the population change in time?

(d) If the initial population satisfies $0 < P(0) < 1$, what will happen to the population after a very long time?

(e) If the initial population satisfies $1 < P(0) < 3$, what will happen to the population after a very long time?

(f) If the initial population satisfies $3 < P(0)$, what will happen to the population after a very long time?

(g) This model for a population's growth is sometimes called "growth with a threshold." Explain why this is an appropriate name.

3. In this problem, we test further what it means for a function to be a solution to a given differential equation.

(a) Consider the differential equation

$$\frac{dy}{dt} = y - t.$$

Determine whether the following functions are solutions to the given differential equation.

(i) $y(t) = t + 1 + 2e^t$

(ii) $y(t) = t + 1$

(iii) $y(t) = t + 2$

(b) When you weigh bananas in a scale at the grocery store, the height h of the bananas is described by the differential equation

$$\frac{d^2h}{dt^2} = -kh$$

where k is the *spring constant*, a constant that depends on the properties of the spring in the scale. After you put the bananas in the scale, you (cleverly) observe that the height of the bananas is given by $h(t) = 4\sin(3t)$. What is the value of the spring constant?

7.2   Qualitative behavior of solutions to DEs

> **Motivating Questions**
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> *In this section, we strive to understand the ideas generated by the following important questions:*
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> - What is a slope field?
>
> - How can we use a slope field to obtain qualitative information about the solutions of a differential equation?
>
> - What are stable and unstable equilibrium solutions of an autonomous differential equation?

Introduction

In earlier work, we have used the tangent line to the graph of a function f at a point a to approximate the values of f near a. The usefulness of this approximation is that we need to know very little about the function; armed with only the value $f(a)$ and the derivative $f'(a)$, we may find the equation of the tangent line and the approximation

$$f(x) \approx f(a) + f'(a)(x - a).$$

Remember that a first-order differential equation gives us information about the derivative of an unknown function. Since the derivative at a point tells us the slope of the tangent line at this point, a differential equation gives us crucial information about the tangent lines to the graph of a solution. We will use this information about the tangent lines to create a *slope field* for the differential equation, which enables us to sketch solutions to initial value problems. Our aim will be to understand the solutions qualitatively. That is, we would like to understand the basic nature of solutions, such as their long-range behavior, without precisely determining the value of a solution at a particular point.

Preview Activity 7.2. Let's consider the initial value problem

$$\frac{dy}{dt} = t - 2, \quad y(0) = 1.$$

(a) Use the differential equation to find the slope of the tangent line to the solution $y(t)$ at $t = 0$. Then use the initial value to find the equation of the tangent line at $t = 0$. Sketch this tangent line over the interval $-0.25 \leq t \leq 0.25$ on the axes provided.
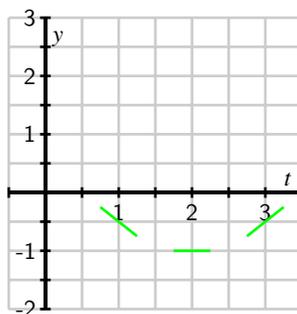
(b) Also shown in the given figure are the tangent lines to the solution $y(t)$ at the points $t = 1, 2$, and 3 (we will see how to find these later). Use the graph to measure the slope of each tangent line and verify that each agrees with the value specified by the differential equation.

(c) Using these tangent lines as a guide, sketch a graph of the solution $y(t)$ over the interval $0 \leq t \leq 3$ so that the lines are tangent to the graph of $y(t)$.

(d) Use the Fundamental Theorem of Calculus to find $y(t)$, the solution to this initial value problem.

(e) Graph the solution you found in (d) on the axes provided, and compare it to the sketch you made using the tangent lines.

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Slope fields

Preview Activity 7.2 shows that we may sketch the solution to an initial value problem if we know an appropriate collection of tangent lines. Because we may use a given differential equation to determine the slope of the tangent line at any point of interest, by plotting a useful collection of these, we can get an accurate sense of how certain solution curves must behave.

Let's continue looking at the differential equation $\dfrac{dy}{dt} = t - 2$. If $t = 0$, this equation says that $dy/dt = 0 - 2 = -2$. Note that this value holds regardless of the value of y. We will therefore sketch tangent lines for several values of y and $t = 0$ with a slope of -2.
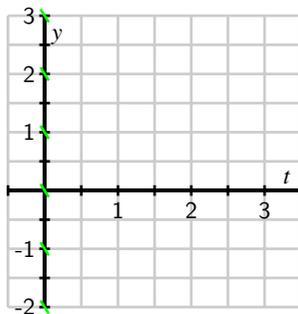
Let's continue in the same way: if $t = 1$, the differential equation tells us that $dy/dt = 1 - 2 = -1$, and this holds regardless of the value of y. We now sketch tangent lines for several values of y and $t = 1$ with a slope of -1.
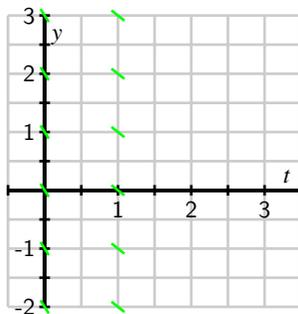

Similarly, we see that when $t = 2$, $dy/dt = 0$ and when $t = 3$, $dy/dt = 1$. We may therefore add to our growing collection of tangent line plots to achieve the next figure.
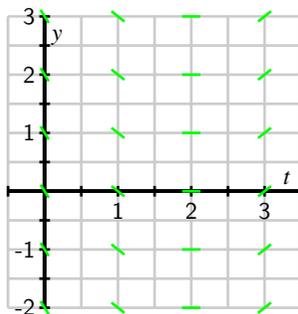

In this figure, you may see the solutions to the differential equation emerge. However, for the sake of clarity, we will add more tangent lines to provide the more complete picture shown below.
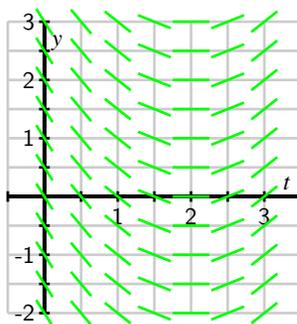
This most recent figure, which is called a *slope field* for the differential equation, allows us to sketch solutions just as we did in the preview activity. Here, we will begin with the initial value $y(0) = 1$ and start sketching the solution by following the tangent line, as shown in the next figure.
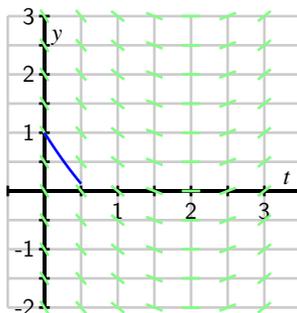
We then continue using this principle: whenever the solution passes through a point at which a tangent line is drawn, that line is tangent to the solution. Doing so leads us to the following sequence of images.
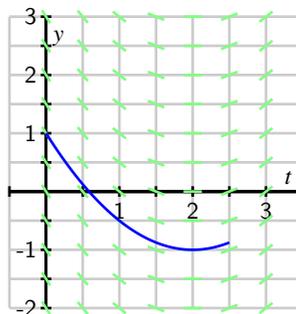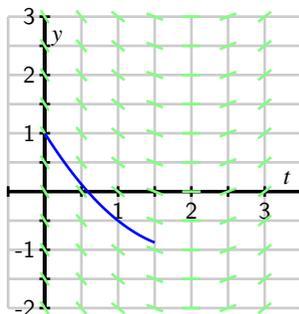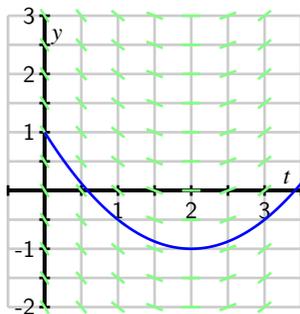
In fact, we may draw solutions for any possible initial value, and doing this for several different initial values for $y(0)$ results in the graphs shown next.
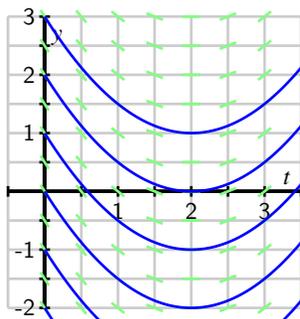
Just as we have done for the most recent example with $\frac{dy}{dt} = t - 2$, we can construct a slope field for any differential equation of interest. The slope field provides us with visual information about how we expect solutions to the differential equation to behave.

Activity 7.4.

Consider the autonomous differential equation

$$\frac{dy}{dt} = -\frac{1}{2}(y - 4).$$

(a) Make a plot of $\frac{dy}{dt}$ versus y on the axes provided. Looking at the graph, for what values of y does y increase and for what values of y does y decrease?
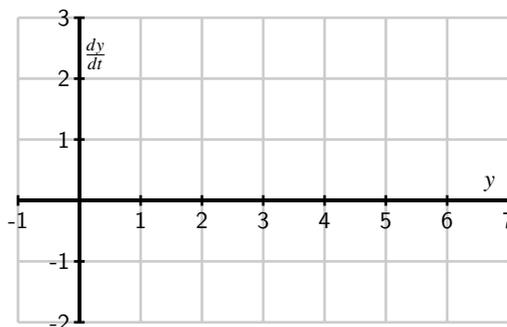
(b) Next, sketch the slope field for this differential equation on the axes provided.
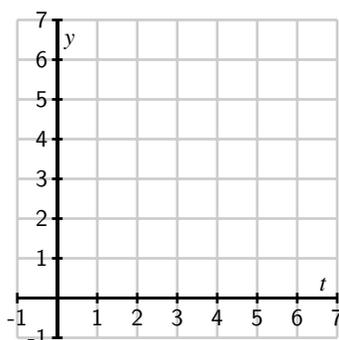
(c) Use your work in (b) to sketch the solutions that satisfy $y(0) = 0$, $y(0) = 2$, $y(0) = 4$ and $y(0) = 6$.

(d) Verify that $y(t) = 4 + 2e^{-t/2}$ is a solution to the given differential equation with the initial value $y(0) = 6$. Compare its graph to the one you sketched in (c).

(e) What is special about the solution where $y(0) = 4$?

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Equilibrium solutions and stability

As our work in Activity 7.4 demonstrates, first-order autonomous solutions may have solutions that are constant. In fact, these are quite easy to detect by inspecting the differential equation $dy/dt = f(y)$: constant solutions necessarily have a zero derivative so $dy/dt = 0 = f(y)$.

For example, in Activity 7.4, we considered the equation

$$\frac{dy}{dt} = f(y) = -\frac{1}{2}(y - 4).$$

Constant solutions are found by setting $f(y) = -\frac{1}{2}(y - 4) = 0$, which we immediately see implies that $y = 4$.

Values of y for which $f(y) = 0$ in an autonomous differential equation $\frac{dy}{dt} = f(y)$ are usually called or *equilibrium solutions* of the differential equation.

Activity 7.5.

Consider the autonomous differential equation

$$\frac{dy}{dt} = -\frac{1}{2}y(y - 4).$$

(a) Make a plot of $\frac{dy}{dt}$ versus y. Looking at the graph, for what values of y does y increase and for what values of y does y decrease?
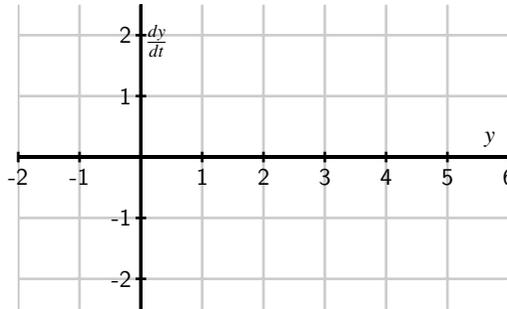
(b) Identify any equilibrium solutions of the given differential equation.

(c) Now sketch the slope field for the given differential equation.
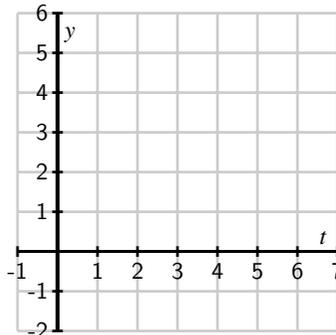
(d) Sketch the solutions to the given differential equation that correspond to initial values $y(0) = -1, 0, 1, \ldots, 5$.

(e) An equilibrium solution $\overline{y}$ is called *stable* if nearby solutions converge to $\overline{y}$. This means that if the initial condition varies slightly from $\overline{y}$, then $\lim_{t \to \infty} y(t) = \overline{y}$.

Conversely, an equilibrium solution $\overline{y}$ is called *unstable* if nearby solutions are pushed away from $\overline{y}$.

Using your work above, classify the equilibrium solutions you found in (b) as either stable or unstable.

(f) Suppose that $y(t)$ describes the population of a species of living organisms and that the initial value $y(0)$ is positive. What can you say about the eventual fate of this population?

(g) Remember that an equilibrium solution $\overline{y}$ satisfies $f(\overline{y}) = 0$. If we graph $dy/dt = f(y)$ as a function of y, for which of the following differential equations is $\overline{y}$ a stable equilibrium and for which is $\overline{y}$ unstable? Why?

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Summary

In this section, we encountered the following important ideas:
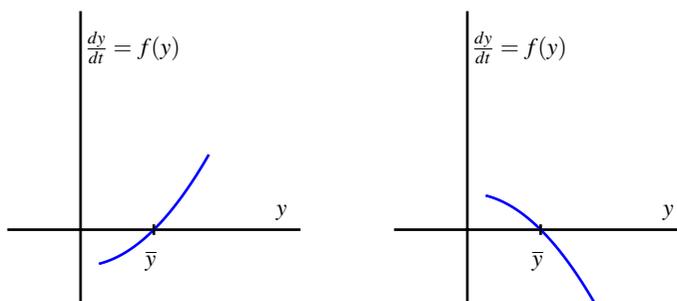
- A slope field is a plot created by graphing the tangent lines of many different solutions to a differential equation.

- Once we have a slope field, we may sketch the graph of solutions by drawing a curve that is always tangent to the lines in the slope field.

- Autonomous differential equations sometimes have constant solutions that we call equilibrium solutions. These may be classified as stable or unstable, depending on the behavior of nearby solutions.

Exercises

1. Consider the differential equation

$$\frac{dy}{dt} = t - y.$$

(a) Sketch a slope field on the plot below:
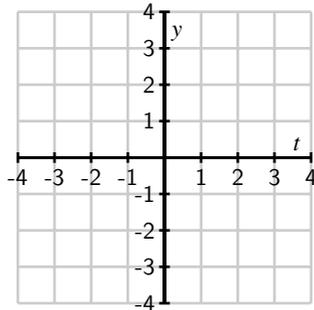
(b) Sketch the solutions whose initial values are $y(0) = -4, -3, \ldots, 4$.

(c) What do your sketches suggest is the solution whose initial value is $y(0) = -1$? Verify that this is indeed the solution to this initial value problem.

(d) By considering the differential equation and the graphs you have sketched, what is the relationship between t and y at a point where a solution has a local minimum?

2. Consider the situation from problem 2 of Section 7.1: Suppose that the population of a particular species is described by the function $P(t)$, where P is expressed in millions. Suppose further that the population's rate of change is governed by the differential equation

$$\frac{dP}{dt} = f(P)$$

where $f(P)$ is the function graphed below.
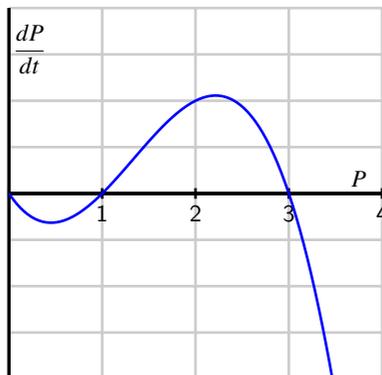
(a) Sketch a slope field for this differential equation. You do not have enough information to determine the actual slopes, but you should have enough information to determine where slopes are positive, negative, zero, large, or small, and hence determine the qualitative behavior of solutions.

(b) Sketch some solutions to this differential equation when the initial population $P(0) > 0$.

(c) Identify any equilibrium solutions to the differential equation and classify them as stable or unstable.

(d) If $P(0) > 1$, what is the eventual fate of the species?

(e) if $P(0) < 1$, what is the eventual fate of the species?

(f) Remember that we referred to this model for population growth as "growth with a threshold." Explain why this characterization makes sense by considering solutions whose inital value is close to 1.

3. The population of a species of fish in a lake is $P(t)$ where P is measured in thousands of fish and t is measured in months. The growth of the population is described by the differential equation

$$\frac{dP}{dt} = f(P) = P(6 - P).$$

(a) Sketch a graph of $f(P) = P(6 - P)$ and use it to determine the equilibrium solutions and whether they are stable or unstable. Write a complete sentence that describes the long-term behavior of the fish population.

(b) Suppose now that the owners of the lake allow fishers to remove 1000 fish from the lake every month (remember that $P(t)$ is measured in *thousands* of fish). Modify the differential equation to take this into account. Sketch the new graph of dP/dt versus P. Determine the new equilibrium solutions and decide whether they are stable or unstable.

(c) Given the situation in part (b), give a description of the long-term behavior of the fish population.

(d) Suppose that fishermen remove h thousand fish per month. How is the differential equation modified?

(e) What is the largest number of fish that can be removed per month without eliminating the fish population? If fish are removed at this maximum rate, what is the eventual population of fish?

4. Let $y(t)$ be the number of thousands of mice that live on a farm; assume time t is measured in years.[2]

(a) The population of the mice grows at a yearly rate that is twenty times the number of mice. Express this as a differential equation.

----

[2]This problem is based on an ecological analysis presented in a research paper by C.S. Hollings: The Components of Predation as Revealed by a Study of Small Mammal Predation of the European Pine Sawfly, *Canadian Entomology* **91**: 283-320.

(b) At some point, the farmer brings C cats to the farm. The number of mice that the cats can eat in a year is

$$M(y) = C\frac{y}{2 + y}$$

thousand mice per year. Explain how this modifies the differential equation that you found in part a).

(c) Sketch a graph of the function $M(y)$ for a single cat $C = 1$ and explain its features by looking, for instance, at the behavior of $M(y)$ when y is small and when y is large.

(d) Suppose that $C = 1$. Find the equilibrium solutions and determine whether they are stable or unstable. Use this to explain the long-term behavior of the mice population depending on the initial population of the mice.

(e) Suppose that $C = 60$. Find the equilibrium solutions and determine whether they are stable or unstable. Use this to explain the long-term behavior of the mice population depending on the initial population of the mice.

(f) What is the smallest number of cats you would need to keep the mice population from growing arbitrarily large?

7.3   Euler's method

> **Motivating Questions**
>
> *In this section, we strive to understand the ideas generated by the following important questions:*
>
> - What is Euler's method and how can we use it to approximate the solution to an initial value problem?
>
> - How accurate is Euler's method?

Introduction

In Section 7.2, we saw how a slope field can be used to sketch solutions to a differential equation. In particular, the slope field is a plot of a large collection of tangent lines to a large number of solutions of the differential equation, and we sketch a single solution by simply following these tangent lines. With a little more thought, we may use this same idea to numerically approximate the solutions of a differential equation.

Preview Activity 7.3. Consider the initial value problem

$$\frac{dy}{dt} = \frac{1}{2}(y + 1), \ \ y(0) = 0.$$

(a) Use the differential equation to find the slope of the tangent line to the solution $y(t)$ at $t = 0$. Then use the given initial value to find the equation of the tangent line at $t = 0$.

(b) Sketch the tangent line on the axes below on the interval $0 \leq t \leq 2$ and use it to approximate $y(2)$, the value of the solution at $t = 2$.
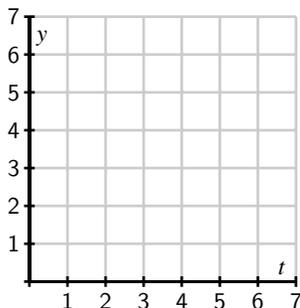
(c) Assuming that your approximation for $y(2)$ is the actual value of $y(2)$, use the differential equation to find the slope of the tangent line to $y(t)$ at $t = 2$. Then, write the equation of the tangent line at $t = 2$.

(d) Add a sketch of this tangent line to your plot on the axes above on the interval $2 \le t \le 4$; use this new tangent line to approximate $y(4)$, the value of the solution at $t = 4$.

(e) Repeat the same step to find an approximation for $y(6)$.

⋈

Euler's Method

Preview Activity 7.3 demonstrates the essence of an algorithm, which is known as Euler's Method, that generates a numerical approximation to the solution of an initial value problem.[3] In this algorithm, we will approximate the solution by taking horizontal steps of a fixed size that we denote by Δt.

Before explaining the algorithm in detail, let's remember how we compute the slope of a line: the slope is the ratio of the vertical change to the horizontal change, as shown in the following figure.
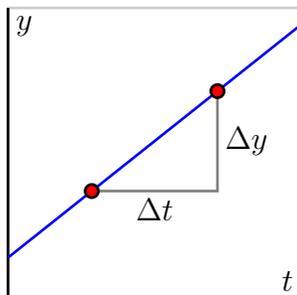
In other words, $m = \frac{\Delta y}{\Delta t}$. Said differently, the vertical change is the product of the slope and the horizontal change:

$$\Delta y = m \Delta t.$$

Suppose that we would like to solve the initial value problem

$$\frac{dy}{dt} = t - y, \ \ y(0) = 1.$$

---

[3]"Euler" is pronounced "Oy-ler." Among other things, Euler is the mathematician credited with the famous number e; if you incorrectly pronounce his name "You-ler," you fail to appreciate his genius and legacy.

While there is an algorithm by which we can find an algebraic formula for the solution to this initial value problem, and we can check that this solution is $y(t) = t - 1 + 2e^{-t}$, we are instead interested in generating an approximate solution by creating a sequence of points (t_i, y_i), where $y_i \approx y(t_i)$. For this first example, we choose $\Delta t = 0.2$.

Since we know that $y(0) = 1$, we will take the initial point to be $(t_0, y_0) = (0, 1)$ and move horizontally by $\Delta t = 0.2$ to the point (t_1, y_1). Therefore, $t_1 = t_0 + \Delta t = 0.2$. The differential equation tells us that the slope of the tangent line at this point is

$$m = \left.\frac{dy}{dt}\right|_{(0,1)} = 0 - 1 = -1.$$

Therefore, if we move along the tangent line by taking a horizontal step of size $\Delta t = 0.2$, we must also move vertically by

$$\Delta y = m\Delta t = -1 \cdot 0.2 = -0.2.$$

We then have the approximation $y(0.2) \approx y_1 = y_0 + \Delta y = 1 - 0.2 = 0.8$. At this point, we have executed one step of Euler's method.

Now we repeat this process: at $(t_1, y_1) = (0.2, 0.8)$, the differential equation tells us that the slope is

$$m = \left.\frac{dy}{dt}\right|_{(0.2, 0.8)} = 0.2 - 0.8 = -0.6.$$

If we move horizontally by Δt to $t_2 = t_1 + \Delta = 0.4$, we must move vertically by

$$\Delta y = -0.6 \cdot 0.2 = -0.12.$$

We consequently arrive at $y_2 = y_1 + \Delta y = 0.8 - 0.12 = 0.68$, which gives $y(0.2) \approx 0.68$. Now we have completed the second step of Euler's method.

If we continue in this way, we may generate the points (t_i, y_i) shown at left in Figure 7.2. In situations where we are able to find a formula for the actual solution $y(t)$, we can graph $y(t)$ to compare it to the points generated by Euler's method, as shown at right in Figure 7.2.

Because we need to generate a large number of points (t_i, y_i), it is convenient to organize the implementation of Euler's method in a table as shown. We begin with the given initial data.
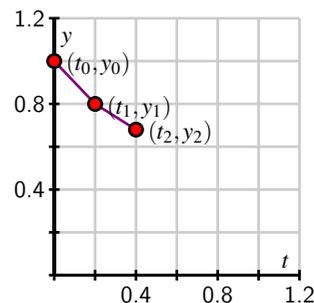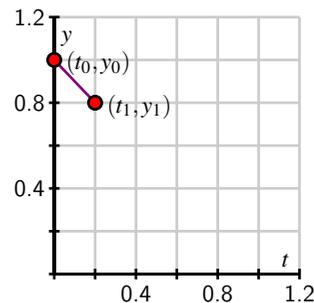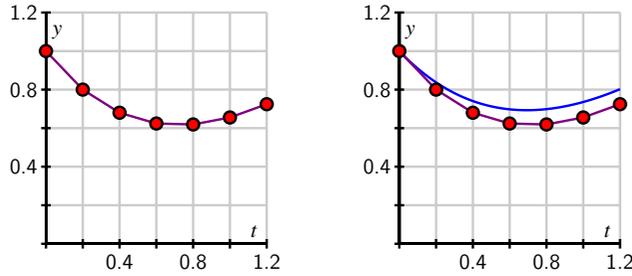
Figure 7.2: At left, the points and piecewise linear approximate solution generated by Euler's method; at right, the approximate solution compared to the exact solution (shown in blue).

t_i	y_i	dy/dt	Δy
0.0000	1.0000		

From here, we compute the slope of the tangent line $m = dy/dt$ using the formula for dy/dt from the differential equation, and then we find Δy, the change in y, using the rule $\Delta y = m\Delta t$.

t_i	y_i	dy/dt	Δy
0.0000	1.0000	-1.0000	-0.2000

Next, we increase t_i by Δt and y_i by Δy to get

t_i	y_i	dy/dt	Δy
0.0000	1.0000	-1.0000	-0.2000
0.2000	0.8000		

and then we simply continue the process for however many steps we decide, eventually generating a table like the one that follows.

t_i	y_i	dy/dt	Δy
0.0000	1.0000	-1.0000	-0.2000
0.2000	0.8000	-0.6000	-0.1200
0.4000	0.6800	-0.2800	-0.0560
0.6000	0.6240	-0.0240	-0.0048
0.8000	0.6192	0.1808	0.0362
1.0000	0.6554	0.3446	0.0689
1.2000	0.7243	0.4757	0.0951

Activity 7.6.

Consider the initial value problem

$$\frac{dy}{dt} = 2t - 1, \ y(0) = 0$$

(a) Use Euler's method with $\Delta t = 0.2$ to approximate the solution at $t_i = 0.2, 0.4, 0.6, 0.8,$ and 1.0. Record your work in the following table, and sketch the points (t_i, y_i) on the following axes provided.

t_i	y_i	dy/dt	Δy
0.0000	0.0000		
0.2000			
0.4000			
0.6000			
0.8000			
1.0000			
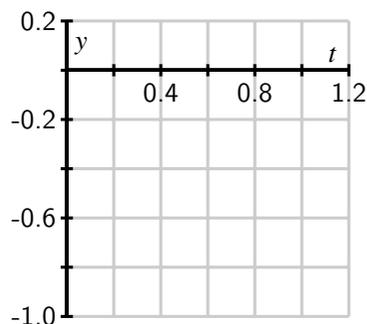
(b) Find the exact solution to the original initial value problem and use this function to find the error in your approximation at each one of the points t_i.

(c) Explain why the value y_5 generated by Euler's method for this initial value problem produces the same value as a left Riemann sum for the definite integral $\int_0^1 (2t - 1) \, dt$.

(d) How would your computations differ if the initial value was $y(0) = 1$? What does this mean about different solutions to this differential equation?

◁

Activity 7.7.

Consider the differential equation $\frac{dy}{dt} = 6y - y^2$.

(a) Sketch the slope field for this differential equation on the axes provided at left below.
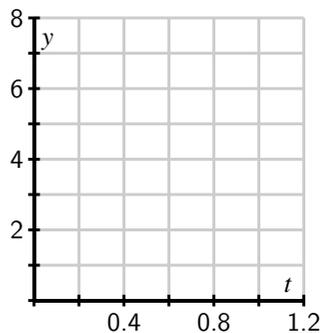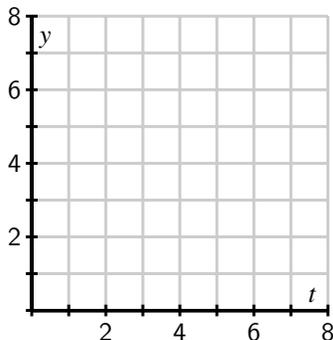
(b) Identify any equilibrium solutions and determine whether they are stable or unstable.

(c) What is the long-term behavior of the solution that satisfies the initial value $y(0) = 1$?

(d) Using the initial value $y(0) = 1$, use Euler's method with $\Delta t = 0.2$ to approximate the solution at $t_i = 0.2, 0.4, 0.6, 0.8$, and 1.0. Sketch the points (t_i, y_i) on the axes provided at right in (a). (Note the different horizontal scale on the two sets of axes.)

t_i	y_i	dy/dt	Δy
0.0	1.0000		
0.2			
0.4			
0.6			
0.8			
1.0			

(e) What happens if we apply Euler's method to approximate the solution with $y(0) = 6$?

◁

The error in Euler's method

Since we are approximating the solutions to an initial value problem using tangent lines, we should expect that the error in the approximation will be less when the step size is smaller. To explore this observation quantitatively, let's consider the initial value problem

$$\frac{dy}{dt} = y, \ y(0) = 1$$

whose solution we can easily find.

Consider the question posed by this initial value problem: "what function do we know that is the same as its own derivative and has value 1 when $t = 0$?" It is not hard to see that the solution is $y(t) = e^t$. We now apply Euler's method to approximate $y(1) = e$ using several values of Δt. These approximations will be denoted by $E_{\Delta t}$, and these estimates provide us a way to see how accurate Euler's Method is.

To begin, we apply Euler's method with a step size of $\Delta t = 0.2$. In that case, we find that $y(1) \approx E_{0.2} = 2.4883$. The error is therefore $y(1) - E_{0.2} = e - 2.4883 \approx 0.2300$.

Repeatedly halving Δt gives the following results, expressed in both tabular and graphical form.

Δt	$E_{\Delta t}$	Error
0.200	2.4883	0.2300
0.100	2.5937	0.1245
0.050	2.6533	0.0650
0.025	2.6851	0.0332
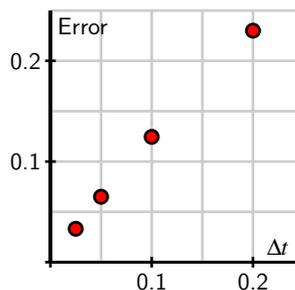
Notice, both numerically and graphically, that the error is roughly halved when Δt is halved. This example illustrates the following general principle.

If Euler's method is to approximate the solution to an initial value problem at a point $\bar{t}$, then the error is proportional to Δt. That is,

$$y(\bar{t}) - E_{\Delta t} \approx K\Delta t$$

for some constant of proportionality K.

Summary

In this section, we encountered the following important ideas:

- Euler's method is an algorithm for approximating the solution to an initial value problem by following the tangent lines while we take horizontal steps across the t-axis.

- If we wish to approximate $y(\bar{t})$ for some fixed $\bar{t}$ by taking horizontal steps of size Δt, then the error in our approximation is proportional to Δt.

Exercises

1. Newton's Law of Cooling says that the rate at which an object, such as a cup of coffee, cools is proportional to the difference in the object's temperature and room temperature. If $T(t)$ is the object's temperature and T_r is room temperature, this law is expressed at

$$\frac{dT}{dt} = -k(T - T_r),$$

where k is a constant of proportionality. In this problem, temperature is measured in degrees Fahrenheit and time in minutes.

   (a) Two calculus students, Alice and Bob, enter a 70° classroom at the same time. Each has a cup of coffee that is 100°. The differential equation for Alice has a constant of proportionality $k = 0.5$, while the constant of proportionality for Bob is $k = 0.1$.

   What is the initial rate of change for Alice's coffee? What is the initial rate of change for Bob's coffee?

   (b) What feature of Alice's and Bob's cups of coffee could explain this difference?

   (c) As the heating unit turns on and off in the room, the temperature in the room is

$$T_r = 70 + 10\sin t.$$

   Implement Euler's method with a step size of $\Delta t = 0.1$ to approximate the temperature of Alice's coffee over the time interval $0 \le t \le 50$. This will most easily be performed using a spreadsheet such as *Excel*. Graph the temperature of her coffee and room temperature over this interval.

   (d) In the same way, implement Euler's method to approximate the temperature of Bob's coffee over the same time interval. Graph the temperature of his coffee and room temperature over the interval.

   (e) Explain the similarities and differences that you see in the behavior of Alice's and Bob's cups of coffee.

2. We have seen that the error in approximating the solution to an initial value problem is proportional to Δt. That is, if $E_{\Delta t}$ is the Euler's method approximation to the solution to an initial value problem at $\bar{t}$, then

$$y(\bar{t}) - E_{\Delta t} \approx K\Delta t$$

for some constant of proportionality K.

In this problem, we will see how to use this fact to improve our estimates, using an idea called *accelerated convergence*.

(a) We will create a new approximation by assuming the error is *exactly* proportional to Δt, according to the formula

$$y(\bar{t}) - E_{\Delta t} = K\Delta t.$$

Using our earlier results from the initial value problem $dy/dt = y$ and $y(0) = 1$ with $\Delta t = 0.2$ and $\Delta t = 0.1$, we have

$$y(1) - 2.4883 = 0.2K$$
$$y(1) - 2.5937 = 0.1K.$$

This is a system of two linear equations in the unknowns $y(1)$ and K. Solve this system to find a new approximation for $y(1)$. (You may remember that the exact value is $y(1) = e = 2.71828\ldots$.)

(b) Use the other data, $E_{0.05} = 2.6533$ and $E_{0.025} = 2.6851$ to do similar work as in (a) to obtain another approximation. Which gives the better approximation? Why do you think this is?

(c) Let's now study the initial value problem

$$\frac{dy}{dt} = t - y, \ y(0) = 0.$$

Approximate $y(0.3)$ by applying Euler's method to find approximations $E_{0.1}$ and $E_{0.05}$. Now use the idea of accelerated convergence to obtain a better approximation. (For the sake of comparison, you want to note that the actual value is $y(0.3) = 0.0408$.)

3. In this problem, we'll modify Euler's method to obtain better approximations to solutions of initial value problems. This method is called the *Improved Euler's method*.

In Euler's method, we walk across an interval of width Δt using the slope obtained from the differential equation at the left endpoint of the interval. Of course, the slope of the

solution will most likely change over this interval. We can improve our approximation by trying to incorporate the change in the slope over the interval.

Let's again consider the initial value problem $dy/dt = y$ and $y(0) = 1$, which we will approximate using steps of width $\Delta t = 0.2$. Our first interval is therefore $0 \leq t \leq 0.2$. At $t = 0$, the differential equation tells us that the slope is 1, and the approximation we obtain from Euler's method is that $y(0.2) \approx y_1 = 1 + 1(0.2) = 1.2$.

This gives us some idea for how the slope has changed over the interval $0 \leq t \leq 0.2$. We know the slope at $t = 0$ is 1, while the slope at $t = 0.2$ is 1.2, trusting in the Euler's method approximation. We will therefore refine our estimate of the initial slope to be the average of these two slopes; that is, we will estimate the slope to be $(1+1.2)/2 = 1.1$. This gives the new approximation $y(1) = y_1 = 1 + 1.1(0.2) = 1.22$.

The first few steps look like this:

t_i	y_i	Slope at (t_{i+1}, y_{i+1})	Average slope
0.0	1.0000	1.2000	1.1000
0.2	1.2200	1.4640	1.3420
0.4	1.4884	1.7861	1.6372
$\vdots$	$\vdots$	$\vdots$	$\vdots$

(a) Continue with this method to obtain an approximation for $y(1) = e$.

(b) Repeat this method with $\Delta t = 0.1$ to obtain a better approximation for $y(1)$.

(c) We saw that the error in Euler's method is proportional to Δt. Using your results from parts (a) and (b), what power of Δt appears to be proportional to the error in the Improved Euler's Method?

7.4 Separable differential equations

> **Motivating Questions**
>
> *In this section, we strive to understand the ideas generated by the following important questions:*
>
> - What is a separable differential equation?
>
> - How can we find solutions to a separable differential equation?
>
> - Are some of the differential equations that arise in applications separable?

Introduction

In Sections 7.2 and 7.3, we have seen several ways to approximate the solution to an initial value problem. Given the frequency with which differential equations arise in the world around us, we would like to have some techniques for finding explicit algebraic solutions of certain initial value problems. In this section, we focus on a particular class of differential equations (called *separable*) and develop a method for finding algebraic formulas for solutions to these equations.

A *separable differential equation* is a differential equation whose algebraic structure permits the variables present to be separated in a particular way. For instance, consider the equation

$$\frac{dy}{dt} = ty.$$

We would like to separate the variables t and y so that all occurrences of t appear on the right-hand side, and all occurrences of y appears on the left and multiply dy/dt. We may do this in the preceding differential equation by dividing both sides by y:

$$\frac{1}{y}\frac{dy}{dt} = t.$$

Note particularly that when we attempt to separate the variables in a differential equation, we require that the left-hand side be a product in which the derivative dy/dt is one term.

Not every differential equation is separable. For example, if we consider the equation

$$\frac{dy}{dt} = t - y,$$

it may seem natural to separate it by writing

$$y + \frac{dy}{dt} = t.$$

As we will see, this will not be helpful since the left-hand side is not a product of a function of y with $\frac{dy}{dt}$.

Preview Activity 7.4. In this preview activity, we explore whether certain differential equations are separable or not, and then revisit some key ideas from earlier work in integral calculus.

(a) Which of the following differential equations are separable? If the equation is separable, write the equation in the revised form $g(y)\frac{dy}{dt} = h(t)$.

1. $\dfrac{dy}{dt} = -3y$.

2. $\dfrac{dy}{dt} = ty - y$.

3. $\dfrac{dy}{dt} = t + 1$.

4. $\dfrac{dy}{dt} = t^2 - y^2$.

(b) Explain why any autonomous differential equation is guaranteed to be separable.

(c) Why do we include the term "$+C$" in the expression

$$\int x\, dx = \frac{x^2}{2} + C?$$

(d) Suppose we know that a certain function f satisfies the equation

$$\int f'(x)\, dx = \int x\, dx.$$

What can you conclude about f?

⋈

Solving separable differential equations

Before we discuss a general approach to solving a separable differential equation, it is instructive to consider an example.

Example 7.1. Find all functions y that are solutions to the differential equation

$$\frac{dy}{dt} = \frac{t}{y^2}.$$

Solution. We begin by separating the variables and writing

$$y^2 \frac{dy}{dt} = t.$$

Integrating both sides of the equation with respect to the independent variable t shows that

$$\int y^2 \frac{dy}{dt} \, dt = \int t \, dt.$$

Next, we notice that the left-hand side allows us to change the variable of antidifferentiation[4] from t to y. In particular, $dy = \frac{dy}{dt} \, dt$, so we now have

$$\int y^2 \, dy = \int t \, dt.$$

This most recent equation says that two families of antiderivatives are equal to one another. Therefore, when we find representative antiderivatives of both sides, we know they must differ by arbitrary constant C. Antidifferentiating and including the integration constant C on the right, we find that

$$\frac{y^3}{3} = \frac{t^2}{2} + C.$$

Again, note that it is not necessary to include an arbitrary constant on both sides of the equation; we know that $y^3/3$ and $t^2/2$ are in the same family of antiderivatives and must therefore differ by a single constant.

Finally, we may now solve the last equation above for y as a function of t, which gives

$$y(t) = \sqrt[3]{\frac{3}{2} t^2 + 3C}.$$

Of course, the term $3C$ on the right-hand side represents 3 times an unknown constant. It is, therefore, still an unknown constant, which we will rewrite as C. We thus conclude that the funtion

$$y(t) = \sqrt[3]{\frac{3}{2} t^2 + C}$$

is a solution to the original differential equation for any value of C.

Notice that because this solution depends on the arbitrary constant C, we have found an infinite family of solutions. This makes sense because we expect to find a unique solution that corresponds to any given initial value.

For example, if we want to solve the initial value problem

$$\frac{dy}{dt} = \frac{t}{y^2}, \quad y(0) = 2,$$

---

[4]This is why we required that the left-hand side be written as a product in which dy/dt is one of the terms.

we know that the solution has the form $y(t) = \sqrt[3]{\frac{3}{2}t^2 + C}$ for some constant C. We therefore must find the appropriate value for C that gives the initial value $y(0) = 2$. Hence,

$$2 = y(0) = \sqrt[3]{\frac{3}{2}0^2 + C} = \sqrt[3]{C},$$

which shows that $C = 2^3 = 8$. The solution to the initial value problem is then

$$y(t) = \sqrt[3]{\frac{3}{2}t^2 + 8}.$$

The strategy of Example 7.1 may be applied to any differential equation of the form $\frac{dy}{dt} = g(y) \cdot h(t)$, and any differential equation of this form is said to be *separable*. We work to solve a separable differential equation by writing

$$\frac{1}{g(y)}\frac{dy}{dt} = h(t),$$

and then integrating both sides with respect to t. After integrating, we strive to solve algebraically for y in order to write y as a function of t.

We consider one more example before doing further exploration in some activities.

Example 7.2. Solve the differential equation

$$\frac{dy}{dt} = 3y.$$

Solution. Following the same strategy as in Example 7.1, we have

$$\frac{1}{y}\frac{dy}{dt} = 3.$$

Integrating both sides with respect to t,

$$\int \frac{1}{y}\frac{dy}{dt}\,dt = \int 3\,dt,$$

and thus

$$\int \frac{1}{y}\,dy = \int 3\,dt.$$

Antidifferentiating and including the integration constant, we find that

$$\ln|y| = 3t + C.$$

Finally, we need to solve for y. Here, one point deserves careful attention. By the definition of the natural logarithm function, it follows that

$$|y| = e^{3t+C} = e^{3t}e^{C}.$$

Since C is an unknown constant, e^{C} is as well, though we do know that it is positive (because e^{x} is positive for any x). When we remove the absolute value in order to solve for y, however, this constant may be either positive or negative. We will denote this updated constant (that accounts for a possible $+$ or $-$) by C to obtain

$$y(t) = Ce^{3t}.$$

There is one more slightly technical point to make. Notice that $y = 0$ is an equilibrium solution to this differential equation. In solving the equation above, we begin by dividing both sides by y, which is not allowed if $y = 0$. To be perfectly careful, therefore, we will typically consider the equilibrium solutions separably. In this case, notice that the final form of our solution captures the equilibrium solution by allowing $C = 0$.

Activity 7.8.

Suppose that the population of a town is growing continuously at an annual rate of 3% per year.

    (a) Let $P(t)$ be the population of the town in year t. Write a differential equation that describes the annual growth rate.

    (b) Find the solutions of this differential equation.

    (c) If you know that the town's population in year 0 is 10,000, find the population $P(t)$.

    (d) How long does it take for the population to double? This time is called the *doubling time*.

    (e) Working more generally, find the doubling time if the annual growth rate is k times the population.

◁

Activity 7.9.

Suppose that a cup of coffee is initially at a temperature of 105° F and is placed in a 75° F room. Newton's law of cooling says that

$$\frac{dT}{dt} = -k(T - 75),$$

where k is a constant of proportionality.

(a) Suppose you measure that the coffee is cooling at one degree per minute at the time the coffee is brought into the room. Use the differential equation to determine the value of the constant k.

(b) Find all the solutions of this differential equation.

(c) What happens to all the solutions as $t \to \infty$? Explain how this agrees with your intuition.

(d) What is the temperature of the cup of coffee after 20 minutes?

(e) How long does it take for the coffee to cool to $80°$?

◁

Activity 7.10.

Solve each of the following differential equations or initial value problems.

(a) $\dfrac{dy}{dt} - (2 - t)y = 2 - t$

(b) $\dfrac{1}{t}\dfrac{dy}{dt} = e^{t^2 - 2y}$

(c) $y' = 2y + 2, \quad y(0) = 2$

(d) $y' = 2y^2, \quad y(-1) = 2$

(e) $\dfrac{dy}{dt} = \dfrac{-2ty}{t^2 + 1}, \quad y(0) = 4$

◁

Summary

In this section, we encountered the following important ideas:

- A separable differential equation is one that may be rewritten with all occurrences of the dependent variable multiplying the derivative and all occurrences of the independent variable on the other side of the equation.

- We may find the solutions to certain separable differential equations by separating variables, integrating with respect to t, and ultimately solving the resulting algebraic equation for y.

- This technique allows us to solve many important differential equations that arise in the world around us. For instance, questions of growth and decay and Newton's Law of Cooling give rise to separable differential equations. Later, we will learn in Section 7.6 that the important logistic differential equation is also separable.

Exercises

1. The mass of a radioactive sample decays at a rate that is proportional to its mass.

   (a) Express this fact as a differential equation for the mass $M(t)$ using k for the constant of proportionality.

   (b) If the initial mass is M_0, find an expression for the mass $M(t)$.

   (c) The *half-life* of the sample is the amount of time required for half of the mass to decay. Knowing that the half-life of Carbon-14 is 5730 years, find the value of k for a sample of Carbon-14.

   (d) How long does it take for a sample of Carbon-14 to be reduced to one-quarter its original mass?

   (e) Carbon-14 naturally occurs in our environment; any living organism takes in Carbon-14 when it eats and breathes. Upon dying, however, the organism no longer takes in Carbon-14.

   Suppose that you find remnants of a pre-historic firepit. By analyzing the charred wood in the pit, you determine that the amount of Carbon-14 is only 30% of the amount in living trees. Estimate the age of the firepit.[5]

2. Consider the initial value problem

   $$\frac{dy}{dt} = -\frac{t}{y}, \quad y(0) = 8$$

   (a) Find the solution of the initial value problem and sketch its graph.

   (b) For what values of t is the solution defined?

   (c) What is the value of y at the last time that the solution is defined?

   (d) By looking at the differential equation, explain why we should not expect to find solutions with the value of y you noted in (c).

3. Suppose that a cylindrical water tank with a hole in the bottom is filled with water. The water, of course, will leak out and the height of the water will decrease. Let $h(t)$ denote the height of the water. A physical principle called *Torricelli's Law* implies that the height decreases at a rate proportional to the square root of the height.

   (a) Express this fact using k as the constant of proportionality.

   (b) Suppose you have two tanks, one with $k = -1$ and another with $k = -10$. What physical differences would you expect to find?

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[5]This approach is the basic idea behind radiocarbon dating.

(c) Suppose you have a tank for which the height decreases at 20 inches per minute when the water is filled to a depth of 100 inches. Find the value of k.

(d) Solve the initial value problem for the tank in part (c), and graph the solution you determine.

(e) How long does it take for the water to run out of the tank?

(f) Is the solution that you found valid for all time t? If so, explain how you know this. If not, explain why not.

4. The *Gompertz equation* is a model that is used to describe the growth of certain populations. Suppose that $P(t)$ is the population of some organism and that

$$\frac{dP}{dt} = -P \ln\left(\frac{P}{3}\right) = -P(\ln P - \ln 3).$$

(a) Sketch a slope field for $P(t)$ over the range $0 \le P \le 6$.

(b) Identify any equilibrium solutions and determine whether they are stable or unstable.

(c) Find the population $P(t)$ assuming that $P(0) = 1$ and sketch its graph. What happens to $P(t)$ after a very long time?

(d) Find the population $P(t)$ assuming that $P(0) = 6$ and sketch its graph. What happens to $P(t)$ after a very long time?

(e) Verify that the long-term behavior of your solutions agrees with what you predicted by looking at the slope field.

7.5   Modeling with differential equations

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How can we use differential equations to describe phenomena in the world around us?

- How can we use differential equations to better understand these phenomena?

Introduction

In our work to date, we have seen several ways that differential equations arise in the natural world, from the growth of a population to the temperature of a cup of coffee. In this section, we will look more closely at how differential equations give us a natural way to describe various phenoma. As we'll see, the key is to focus on understanding the different factors that cause a quantity to change.

Preview Activity 7.5. Any time that the rate of change of a quantity is related to the amount of a quantity, a differential equation naturally arises. In the following two problems, we see two such scenarios; for each, we want to develop a differential equation whose solution is the quantity of interest.

(a) Suppose you have a bank account in which money grows at an annual rate of 3%.

    (i) If you have $10,000 in the account, at what rate is your money growing?

    (ii) Suppose that you are also withdrawing money from the account at $1,000 per year. What is the rate of change in the amount of money in the account? What are the units on this rate of change?

(b) Suppose that a water tank holds 100 gallons and that a salty solution, which contains 20 grams of salt in every gallon, enters the tank at 2 gallons per minute.

    (i) How much salt enters the tank each minute?

    (ii) Suppose that initially there are 300 grams of salt in the tank. How much salt is in each gallon at this point in time?

    (iii) Finally, suppose that evenly mixed solution is pumped out of the tank at the rate of 2 gallons per minute. How much salt leaves the tank each minute?

    (iv) What is the total rate of change in the amount of salt in the tank?

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Developing a differential equation

Preview activity 7.5 demonstrates the kind of thinking we will be doing in this section. In each of the two examples we considered, there is a quantity, such as the amount of money in the bank account or the amount of salt in the tank, that is changing due to several factors. The governing differential equation results from the total rate of change being the difference between the rate of increase and the rate of decrease.

Example 7.3. In the Great Lakes region, rivers flowing into the lakes carry a great deal of pollution in the form of small pieces of plastic averaging 1 millimeter in diameter. In order to understand how the amount of plastic in Lake Michigan is changing, construct a model for how this type pollution has built up in the lake.

Solution.

First, some basic facts about Lake Michigan.

- The volume of the lake is $5 \cdot 10^{12}$ cubic meters.

- Water flows into the lake at a rate of $5 \cdot 10^{10}$ cubic meters per year. It flows out of the lake at the same rate.

- Each cubic meter flowing into the lake contains roughly $3 \cdot 10^{-8}$ cubic meters of plastic pollution.

Let's denote the amount of pollution in the lake by $P(t)$, where P is measured in cubic meters of plastic and t in years. Our goal is to describe the rate of change of this function; in other words, we want to develop a differential equation describing $P(t)$.

First, we will measure how $P(t)$ increases due to pollution flowing into the lake. We know that $5 \cdot 10^{10}$ cubic meters of water enters the lake every year and each cubic meter of water contains $3 \cdot 10^{-8}$ cubic meters of pollution. Therefore, pollution enters the lake at the rate of

$$\left(5 \cdot 10^{10} \frac{m^3 \text{ water}}{\text{year}}\right) \cdot \left(3 \cdot 10^{-8} \frac{m^3 \text{ plastic}}{m^3 \text{ water}}\right) = 1.5 \cdot 10^3 \quad \text{cubic m of plastic per year.}$$

Second, we will measure how $P(t)$ decreases due to pollution flowing out of the lake. If the total amount of pollution is P cubic meters and the volume of Lake Michigan is $5 \cdot 10^{12}$ cubic meters, then the concentration of plastic pollution in Lake Michigan is

$$\frac{P}{5 \cdot 10^{12}} \quad \text{cubic meters of plastic per cubic meter of water.}$$

Since $5 \cdot 10^{10}$ cubic meters of water flow out each year[6], then the plastic pollution leaves the lake at the rate of

$$\left(\frac{P}{5 \cdot 10^{12}} \frac{m^3 \text{ plastic}}{m^3 \text{ water}} \right) \cdot \left(5 \cdot 10^{10} \frac{m^3 \text{ water}}{\text{year}} \right) = \frac{P}{100} \quad \text{cubic meters of plastic per year.}$$

The total rate of change of P is thus the difference between the rate at which pollution enters the lake minus the rate at which pollution leaves the lake; that is,

$$\begin{aligned} \frac{dP}{dt} &= 1.5 \cdot 10^3 - \frac{P}{100} \\ &= \frac{1}{100}(1.5 \cdot 10^5 - P). \end{aligned}$$

We have now found a differential equation that describes the rate at which the amount of pollution is changing. To better understand the behavior of $P(t)$, we now apply some of the techniques we have recently developed.

Since this is an autonomous differential equation, we can sketch dP/dt as a function of P and then construct a slope field, as shown in Figure 7.3.
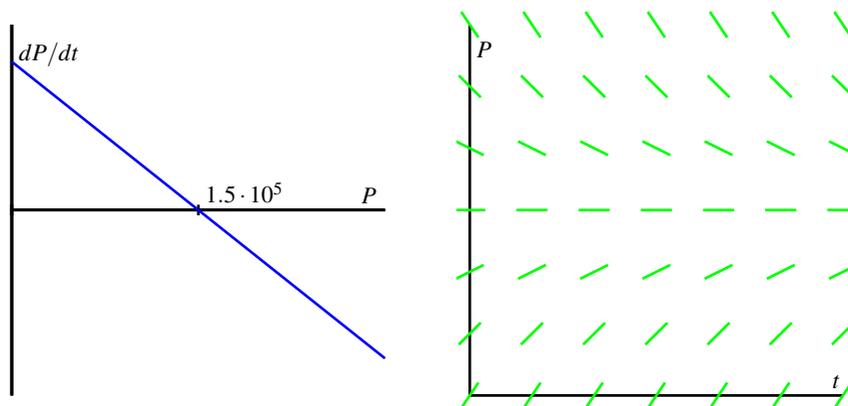

Figure 7.3: Plots of $\frac{dP}{dt}$ vs. P and the slope field for the differential equation $\frac{dP}{dt} = \frac{1}{100}(1.5 \cdot 10^5 - P)$.

These plots both show that $P = 1.5 \cdot 10^5$ is a stable equilibrium. Therefore, we should expect that the amount of pollution in Lake Michigan will stabilize near $1.5 \cdot 10^5$ cubic meters of pollution.

Next, assuming that there is initially no pollution in the lake, we will solve the initial

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[6]and we assume that each cubic meter of water that flows out carries with it the plastic pollution it contains

value problem

$$\frac{dP}{dt} = \frac{1}{100}(1.5 \cdot 10^5 - P), \ P(0) = 0.$$

Separating variables, we find that

$$\frac{1}{1.5 \cdot 10^5 - P}\frac{dP}{dt} = \frac{1}{100}.$$

Integrating with respect to t, we have

$$\int \frac{1}{1.5 \cdot 10^5 - P}\frac{dP}{dt}\, dt = \int \frac{1}{100}\, dt,$$

and thus changing variables on the left and antidifferentiating on both sides, we find that

$$\int \frac{dP}{1.5 \cdot 10^5 - P} = \int \frac{1}{100}\, dt$$

$$-\ln|1.5 \cdot 10^5 - P| = \frac{1}{100}t + C$$

Finally, multiplying both sides by -1 and using the definition of the logarithm, we find that

$$1.5 \cdot 10^5 - P = Ce^{-t/100}. \tag{7.1}$$

This is a good time to determine the constant C. Since $P = 0$ when $t = 0$, we have

$$1.5 \cdot 10^5 - 0 = Ce^0 = C.$$

In other words, $C = 1.5 \cdot 10^5$.

Using this value of C in Equation (7.1) and solving for P, we arrive at the solution

$$P(t) = 1.5 \cdot 10^5(1 - e^{-t/100}).$$

Superimposing the graph of P on the slope field we saw in Figure 7.3, we see, as shown in Figure 7.4 We see that, as expected, the amount of plastic pollution stabilizes around $1.5 \cdot 10^5$ cubic meters.

There are many important lessons to learn from Example 7.3. Foremost is how we can develop a differential equation by thinking about the "total rate = rate in - rate out" model. In addition, we note how we can bring together all of our available understanding (plotting $\frac{dP}{dt}$ vs. P, creating a slope field, solving the differential equation) to see how the differential equation describes the behavior of a changing quantity.

Of course, we can also explore what happens when certain aspects of the problem change. For instance, let's suppose we are at a time when the plastic pollution entering
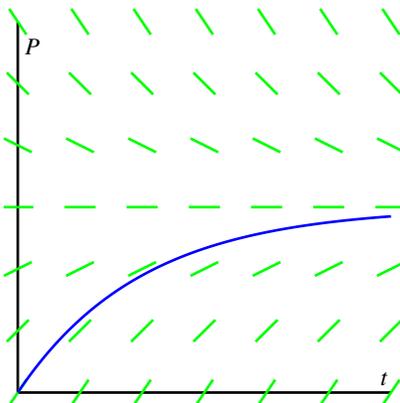
Figure 7.4: The solution $P(t)$ and the slope field for the differential equation $\frac{dP}{dt} = \frac{1}{100}(1.5 \cdot 10^5 - P)$.

Lake Michigan has stabilized at $1.5 \cdot 10^5$ cubic meters, and that new legislation is passed to prevent this type of pollution entering the lake. So, there is no longer any inflow of plastic pollution to the lake. How does the amount of plastic pollution in Lake Michigan now change? For example, how long does it take for the amount of plastic pollution in the lake to halve?

Restarting the problem at time $t = 0$, we now have the modified initial value problem

$$\frac{dP}{dt} = -\frac{1}{100}P, \ \ P(0) = 1.5 \cdot 10^5.$$

It is a straightforward and familiar exercise to find that the solution to this equation is $P(t) = 1.5 \cdot 10^5 e^{-t/100}$. The time that it takes for half of the pollution to flow out of the lake is given by T where $P(T) = 0.75 \cdot 10^5$. Thus, we must solve the equation

$$0.75 \cdot 10^5 = 1.5 \cdot 10^5 e^{-T/100},$$

or

$$\frac{1}{2} = e^{-T/100}.$$

It follows that

$$T = -100 \ln\left(\frac{1}{2}\right) \approx 69.3 \quad \text{years.}$$

In the upcoming activities, we explore some other natural settings in which differential equation model changing quantities.

Activity 7.11.

Suppose you have a bank account that grows by 5% every year. Let $A(t)$ be the amount of money in the account in year t.

(a) What is the rate of change of A with respect to t?

(b) Suppose that you are also withdrawing $10,000 per year. Write a differential equation that expresses the total rate of change of A.

(c) Sketch a slope field for this differential equation, find any equilibrium solutions, and identify them as either stable or unstable. Write a sentence or two that describes the significance of the stability of the equilibrium solution.

(d) Suppose that you initially deposit $100,000 into the account. How long does it take for you to deplete the account?

(e) What is the smallest amount of money you would need to have in the account to guarantee that you never deplete the money in the account?

(f) If your initial deposit is $300,000, how much could you withdraw every year without depleting the account?

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Activity 7.12.

A dose of morphine is absorbed from the bloodstream of a patient at a rate proportional to the amount in the bloodstream.

(a) Write a differential equation for $M(t)$, the amount of morphine in the patient's bloodstream, using k as the constant proportionality.

(b) Assuming that the initial dose of morphine is M_0, solve the initial value problem to find $M(t)$. Use the fact that the half-life for the absorption of morphine is two hours to find the constant k.

(c) Suppose that a patient is given morphine intravenously at the rate of 3 milligrams per hour. Write a differential equation that combines the intravenous administration of morphine with the body's natural absorption.

(d) Find any equilibrium solutions and determine their stability.

(e) Assuming that there is initially no morphine in the patient's bloodstream, solve the initial value problem to determine $M(t)$. What happens to $M(t)$ after a very long time?

(f) To what rate should a doctor reduce the intravenous rate so that there is eventually 7 milligrams of morphine in the patient's bloodstream?

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Summary

In this section, we encountered the following important ideas:

- Differential equations arise in a situation when we understand how various factors cause a quantity to change.

- We may use the tools we have developed so far—slope fields, Euler's methods, and our method for solving separable equations—to understand a quantity described by a differential equation.

Exercises

1. Congratulations, you just won the lottery! In one option presented to you, you will be paid one million dollars a year for the next 25 years. You can deposit this money in an account that will earn 5% each year.

   (a) Set up a differential equation that describes the rate of change in the amount of money in the account. Two factors cause the amount to grow—first, you are depositing one millon dollars per year and second, you are earning 5% interest.

   (b) If there is no amount of money in the account when you open it, how much money will you have in the account after 25 years?

   (c) The second option presented to you is to take a lump sum of 10 million dollars, which you will deposit into a similar account. How much money will you have in that account after 25 years?

   (d) Do you prefer the first or second option? Explain your thinking.

   (e) At what time does the amount of money in the account under the first option overtake the amount of money in the account under the second option?

2. When a skydiver jumps from a plane, gravity causes her downward velocity to increase at the rate of $g \approx 9.8$ meters per second squared. At the same time, wind resistance causes her velocity to decrease at a rate proportional to the velocity.

   (a) Using k to represent the constant of proportionality, write a differential equation that describes the rate of change of the skydiver's velocity.

   (b) Find any equilibrium solutions and decide whether they are stable or unstable. Your result should depend on k.

   (c) Suppose that the initial velocity is zero. Find the velocity $v(t)$.

   (d) A typical terminal velocity for a skydiver falling face down is 54 meters per second. What is the value of k for this skydiver?

(e) How long does it take to reach 50% of the terminal velocity?

3. During the first few years of life, the rate at which a baby gains weight is proportional to the reciprocal of its weight.

   (a) Express this fact as a differential equation.

   (b) Suppose that a baby weighs 8 pounds at birth and 9 pounds one month later. How much will he weigh at one year?

   (c) Do you think this is a realistic model for a long time?

4. Suppose that you have a water tank that holds 100 gallons of water. A briny solution, which contains 20 grams of salt per gallon, enters the tank at the rate of 3 gallons per minute.

   At the same time, the solution is well mixed, and water is pumped out of the tank at the rate of 3 gallons per minute.

   (a) Since 3 gallons enters the tank every minute and 3 gallons leaves every minute, what can you conclude about the volume of water in the tank.

   (b) How many grams of salt enters the tank every minute?

   (c) Suppose that $S(t)$ denotes the number of grams of salt in the tank in minute t. How many grams are there in each gallon in minute t?

   (d) Since water leaves the tank at 3 gallons per minute, how many grams of salt leave the tank each minute?

   (e) Write a differential equation that expresses the total rate of change of S.

   (f) Identify any equilibrium solutions and determine whether they are stable or unstable.

   (g) Suppose that there is initially no salt in the tank. Find the amount of salt $S(t)$ in minute t.

   (h) What happens to $S(t)$ after a very long time? Explain how you could have predicted this only knowing how much salt there is in each gallon of the briny solution that enters the tank.

7.6 Population Growth and the Logistic Equation

Motivating Questions

In this section, we strive to understand the ideas generated by the following important questions:

- How can we use differential equations to realistically model the growth of a population?

- How can we assess the accuracy of our models?

Introduction

The growth of the earth's population is one of the pressing issues of our time. Will the population continue to grow? Or will it perhaps level off at some point, and if so, when? In this section, we will look at two ways in which we may use differential equations to help us address questions such as these.

Before we begin, let's consider again two important differential equations that we have seen in earlier work this chapter.

Preview Activity 7.6. Recall that one model for population growth states that a population grows at a rate proportional to its size.

(a) We begin with the differential equation

$$\frac{dP}{dt} = \frac{1}{2}P.$$

Sketch a slope field below as well as a few typical solutions on the axes provided.
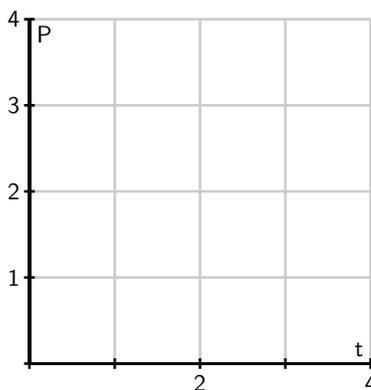
(b) Find all equilibrium solutions of the equation $\frac{dP}{dt} = \frac{1}{2}P$ and classify them as stable or unstable.

(c) If $P(0)$ is positive, describe the long-term behavior of the solution to $\frac{dP}{dt} = \frac{1}{2}P$.

(d) Let's now consider a modified differential equation given by

$$\frac{dP}{dt} = \frac{1}{2}P(3 - P).$$

As before, sketch a slope field as well as a few typical solutions on the following axes provided.
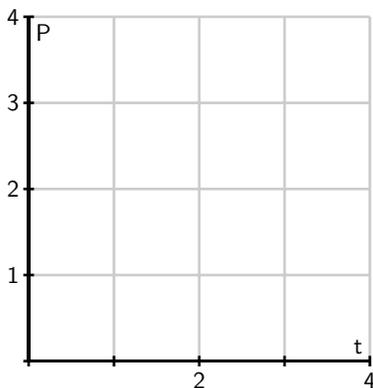
(e) Find any equilibrium solutions and classify them as stable or unstable.

(f) If $P(0)$ is positive, describe the long-term behavior of the solution.

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The earth's population

We will now begin studying the earth's population. To get started, here are some data for the earth's population in recent years that we will use in our investigations.

Year	Population (in billions)
1998	5.932
1999	6.008
2000	6.084
2001	6.159
2002	6.234
2005	6.456
2006	6.531
2007	6.606
2008	6.681
2009	6.756
2010	6.831

Activity 7.13.

Our first model will be based on the following assumption:

The rate of change of the population is proportional to the population.

On the face of it, this seems pretty reasonable. When there is a relatively small number of people, there will be fewer births and deaths so the rate of change will be small. When there is a larger number of people, there will be more births and deaths so we expect a larger rate of change.

If $P(t)$ is the population t years after the year 2000, we may express this assumption as

$$\frac{dP}{dt} = kP$$

where k is a constant of proportionality.

(a) Use the data in the table to estimate the derivative $P'(0)$ using a central difference. Assume that $t = 0$ corresponds to the year 2000.

(b) What is the population $P(0)$?

(c) Use these two facts to estimate the constant of proportionality k in the differential equation.

(d) Now that we know the value of k, we have the initial value problem

$$\frac{dP}{dt} = kP, \ \ P(0) = 6.084.$$

Find the solution to this initial value problem.

(e) What does your solution predict for the population in the year 2010? Is this close to the actual population given in the table?

(f) When does your solution predict that the population will reach 12 billion?

(g) What does your solution predict for the population in the year 2500?

(h) Do you think this is a reasonable model for the earth's population? Why or why not? Explain your thinking using a couple of complete sentences.

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Our work in Activity 7.13 shows that that the exponential model is fairly accurate for years relatively close to 2000. However, if we go too far into the future, the model predicts increasingly large rates of change, which causes the population to grow arbitrarily large. This does not make much sense since it is unrealistic to expect that the earth would be able to support such a large population.

The constant k in the differential equation has an important interpretation. Let's rewrite the differential equation $\frac{dP}{dt} = kP$ by solving for k, so that we have

$$k = \frac{dP/dt}{P}.$$

Viewed in this light, k is the ratio of the rate of change to the population; in other words, it is the contribution to the rate of change from a single person. We call this the *per capita growth rate*.

In the exponential model we introduced in Activity 7.13, the per capita growth rate is constant. In particular, we are assuming that when the population is large, the per capita growth rate is the same as when the population is small. It is natural to think that the per capita growth rate should decrease when the population becomes large, since there will not be enough resources to support so many people. In other words, we expect that a more realistic model would hold if we assume that the per capita growth rate depends on the population P.

In the previous activity, we computed the per capita growth rate in a single year by computing k, the quotient of $\frac{dP}{dt}$ and P (which we did for $t = 0$). If we return data and compute the per capita growth rate over a range of years, we generate the data shown in Figure 7.5, which shows how the per capita growth rate is a function of the population, P. From the data, we see that the per capita growth rate appears to decrease as the population increases. In fact, the points seem to lie very close to a line, which is shown at two different scales in Figure 7.6. Looking at this line carefully, we can find its equation to be

$$\frac{dP/dt}{P} = 0.025 - 0.002P.$$

If we multiply both sides by P, we arrive at the differential equation
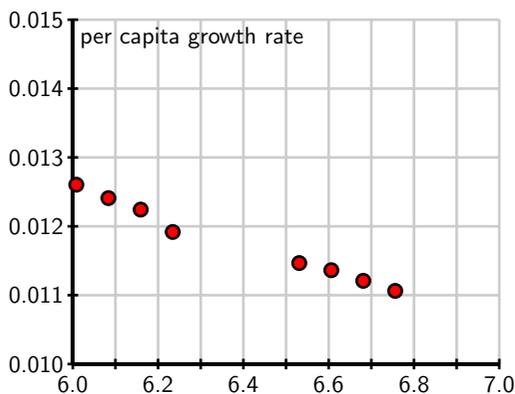
$$\frac{dP}{dt} = P(0.025 - 0.002P).$$

Figure 7.5: A plot of per capita growth rate vs. population P.

Graphing the dependence of dP/dt on the population P, we see that this differential equation demonstrates a quadratic relationship between $\frac{dP}{dt}$ and P, as shown in Figure 7.7. The equation $\frac{dP}{dt} = P(0.025 - 0.002P)$ is an example of the *logistic equation*, and is the second model for population growth that we will consider. We have reason to believe that it will be more realistic since the per capita growth rate is a decreasing function of the population.

Indeed, the graph in Figure 7.7 shows that there are two equilibrium solutions, $P = 0$, which is unstable, and $P = 12.5$, which is a stable equilibrium. The graph shows that any solution with $P(0) > 0$ will eventually stabilize around 12.5. In other words, our model predicts the world's population will eventually stabilize around 12.5 billion.

A prediction for the long-term behavior of the population is a valuable conclusion to draw from our differential equation. We would, however, like to answer some quantitative questions. For instance, how long will it take to reach a population of 10 billion? To determine this, we need to find an explicit solution of the equation.

Solving the logistic differential equation

Since we would like to apply the logistic model in more general situations, we state the logistic equation in its more general form,

$$\frac{dP}{dt} = kP(N - P). \tag{7.2}$$

The equilibrium solutions here are when $P = 0$ and $1 - \frac{P}{N} = 0$, which shows that $P = N$. The equilibrium at $P = N$ is called the *carrying capacity* of the population for it represents the stable population that can be sustained by the environment.
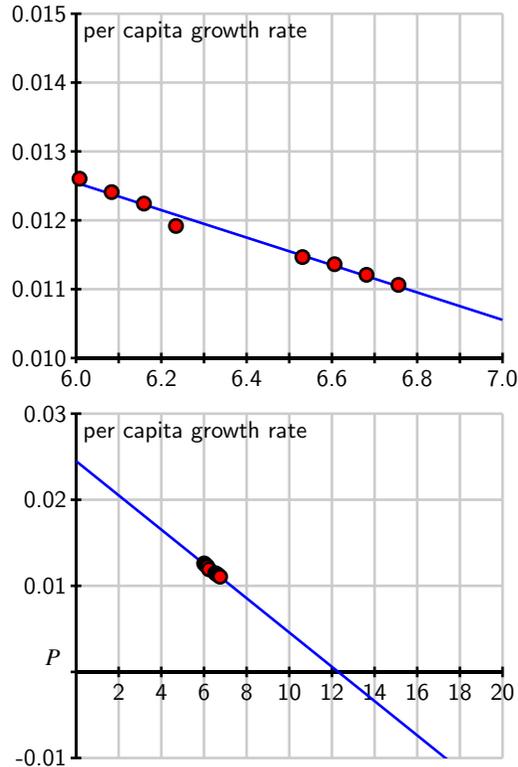
Figure 7.6: The line that approximates per capita growth as a function of population, P.

We now solve the logistic equation (7.2). The equation is separable, so we separate the variables

$$\frac{1}{P(N-P)}\frac{dP}{dt} = k,$$

and integrate to find that

$$\int \frac{1}{P(N-P)}\, dP = \int k\, dt.$$

To find the antiderivative on the left, we use the partial fraction decomposition

$$\frac{1}{P(N-P)} = \frac{1}{N}\left[\frac{1}{P} + \frac{1}{N-P}\right].$$

Now we are ready to integrate, with

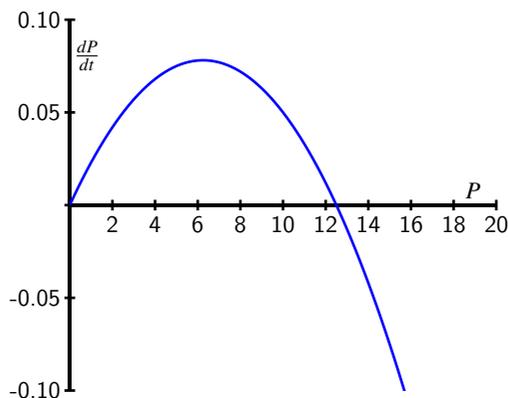$$\int \frac{1}{N}\left[\frac{1}{P} + \frac{1}{N-P}\right]\, dP = \int k\, dt.$$

Figure 7.7: A plot of $\frac{dP}{dt}$ vs. P for the differential equation $\frac{dP}{dt} = P(0.025 - 0.002P)$.

On the left, observe that N is constant, so we can remove the factor of $\frac{1}{N}$ and antidifferentiate to find that

$$\frac{1}{N}(\ln |P| - \ln |N - P|) = kt + C.$$

Multiplying both sides of this last equation by N and using an important rule of logarithms, we next find that

$$\ln \left| \frac{P}{N - P} \right| = kNt + C.$$

From the definition of the logarithm, replacing e^C with C, and letting C absorb the absolute value signs, we now know that

$$\frac{P}{N - P} = Ce^{kNt}.$$

At this point, all that remains is to determine C and solve algebraically for P.

If the initial population is $P(0) = P_0$, then it follows that $C = \frac{P_0}{N - P_0}$, so

$$\frac{P}{N - P} = \frac{P_0}{N - P_0}e^{kNt}.$$

We will solve this most recent equation for P by multiplying both sides by $(N - P)(N - P_0)$ to obtain

$$
\begin{aligned}
P(N - P_0) &= P_0(N - P)e^{kNt} \\
&= P_0 Ne^{kNt} - P_0 Pe^{kNt}.
\end{aligned}
$$

Swapping the left and right sides, expanding, and factoring, it follows that

$$
\begin{aligned}
P_0 N e^{kNt} &= P(N - P_0) + P_0 P e^{kNt} \\
&= P(N - P_0 + P_0 e^{kNt}).
\end{aligned}
$$

Dividing to solve for P, we see that

$$
P = \frac{P_0 N e^{kNt}}{N - P_0 + P_0 e^{kNt}}.
$$

Finally, we choose to multiply the numerator and denominator by $\frac{1}{P_0} e^{-kNt}$ to obtain

$$
P(t) = \frac{N}{\left(\frac{N-P_0}{P_0}\right) e^{-kNt} + 1}.
$$

While that was a lot of algebra, notice the result: we have found an explicit solution to the initial value problem

$$
\frac{dP}{dt} = kP(N - P), \ \ P(0) = P_0,
$$

and that solution is

$$
P(t) = \frac{N}{\left(\frac{N-P_0}{P_0}\right) e^{-kNt} + 1}. \tag{7.3}
$$

For the logistic equation describing the earth's population that we worked with earlier in this section, we have

$$
k = 0.002, \quad N = 12.5, \quad \text{and} \quad P_0 = 6.084.
$$

This gives the solution

$$
P(t) = \frac{12.5}{1.0546 e^{-0.025t} + 1},
$$

whose graph is shown in Figure 7.8 Notice that the graph shows the population leveling off at 12.5 billion, as we expected, and that the population will be around 10 billion in the year 2050. These results, which we have found using a relatively simple mathematical model, agree fairly well with predictions made using a much more sophisticated model developed by the United Nations.

The logistic equation is useful in other situations, too, as it is good for modeling any situation in which limited growth is possible. For instance, it could model the spread of a flu virus through a population contained on a cruise ship, the rate at which a rumor spreads within a small town, or the behavior of an animal population on an island. Again, it is important to realize that through our work in this section, we have completely solved the logistic equation, regardless of the values of the constants N, k, and P_0. Anytime we
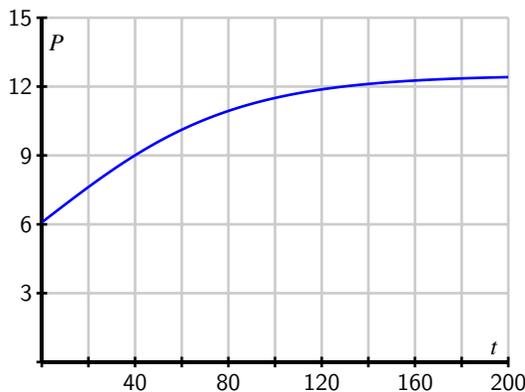
Figure 7.8: The solution to the logistic equation modeling the earth's population.
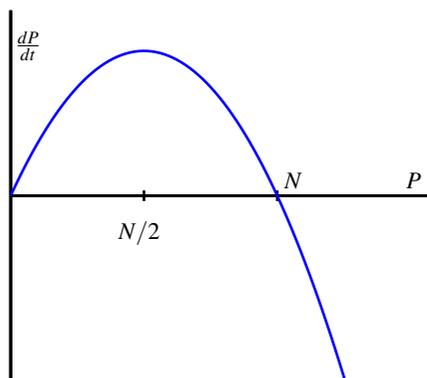
encounter a logistic equation, we can apply the formula we found in Equation (7.3).

Activity 7.14.

Consider the logistic equation

$$\frac{dP}{dt} = kP(N - P)$$

with the graph of $\frac{dP}{dt}$ vs. P shown below.

(a) At what value of P is the rate of change greatest?

(b) Consider the model for the earth's population that we created. At what value of P is the rate of change greatest? How does that compare to the population in recent years?

(c) According to the model we developed, what will the population be in the year 2100?

(d) According to the model we developed, when will the population reach 9 billion?

(e) Now consider the general solution to the general logistic initial value problem that we found, given by

$$P(t) = \frac{N}{\left(\frac{N-P_0}{P_0}\right)e^{-kNt} + 1}.$$

Verify algebraically that $P(0) = P_0$ and that $\lim_{t \to \infty} P(t) = N$.

◁

Summary

In this section, we encountered the following important ideas:

- If we assume that the rate of growth of a population is proportional to the population, we are led to a model in which the population grows without bound and at a rate that grows without bound.

- By assuming that the per capita growth rate decreases as the population grows, we are led to the logistic model of population growth, which predicts that the population will eventually stabilize at the carrying capacity.

Exercises

1. The logistic equation may be used to model how a rumor spreads through a group of people. Suppose that $p(t)$ is the fraction of people that have heard the rumor on day t. The equation

$$\frac{dp}{dt} = 0.2p(1-p)$$

describes how p changes. Suppose initially that one-tenth of the people have heard the rumor; that is, $p(0) = 0.1$.

(a) What happens to $p(t)$ after a very long time?

(b) Determine a formula for the function $p(t)$.

(c) At what time is p changing most rapidly?

(d) How long does it take before 80% of the people have heard the rumor?

2. Suppose that $b(t)$ measures the number of bacteria living in a colony in a Petri dish, where b is measured in thousands and t is measured in days. One day, you measure that there are 6,000 bacteria and the per capita growth rate is 3. A few days later, you measure that there are 9,000 bacteria and the per capita growth rate is 2.

(a) Assume that the per capita growth rate $\frac{db/dt}{b}$ is a linear function of b. Use the measurements to find this function and write a logistic equation to describe $\frac{db}{dt}$.

(b) What is the carrying capacity for the bacteria?

(c) At what population is the number of bacteria increasing most rapidly?

(d) If there are initially 1,000 bacteria, how long will it take to reach 80% of the carrying capacity?

3. Suppose that the population of a species of fish is controlled by the logistic equation

$$\frac{dP}{dt} = 0.1P(10 - P),$$

where P is measured in thousands of fish and t is measured in years.

(a) What is the carrying capacity of this population?

(b) Suppose that a long time has passed and that the fish population is stable at the carrying capacity. At this time, humans begin harvesting 20% of the fish every year. Modify the differential equation by adding a term to incorporate the harvesting of fish.

(c) What is the new carrying capacity?

(d) What will the fish population be one year after the harvesting begins?

(e) How long will it take for the population to be within 10% of the carrying capacity?